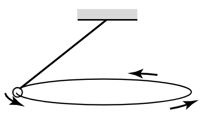
Uniform circular mo ion

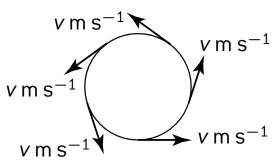
MECHANICS OF CIRCULAR MOTION

The phrase 'uni orm circular motion' is used to describe an object that is going around a circle at constant speed. Most o the time this also means that the circle is horizontal. An example o uni orm circular motion would be the motion o a small mass on the end o a string as shown below.



Example o uni orm circular motion

It is important to remember that even though the speed o the object is constant, its direction is changing all the time.



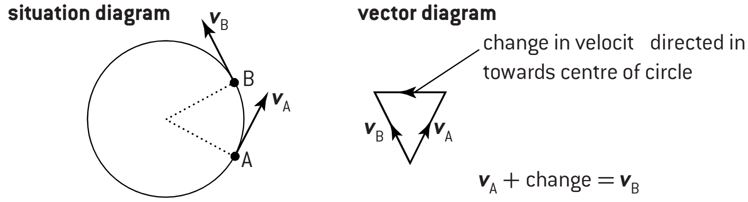
Circular motion – the direction o motion is changing all the time

This constantly changing direction means that the velocity o the object is constantly changing. The word 'acceleration' is used whenever an object's velocity changes. This means that an object in uni orm circular motion MUST be accelerating even i the speed is constant.

The acceleration o a particle travelling in circular motion is called the **centripetal acceleration**. The orce needed to cause the centripetal acceleration is called the **centripetal force**.

MATHEMATICS OF CIRCULAR MOTION

The diagram below allows us to work out the direction o the centripetal acceleration – which must also be the direction o the centripetal orce. This direction is constantly changing.



The object is shown moving between two points A and B on a horizontal circle. Its velocity has changed rom $_A$ to $_B$. The magnitude o velocity is always the same, but the direction has changed. Since velocities are vector quantities we need to use vector mathematics to work out the average change in velocity. This vector diagram is also shown above.

In this example, the direction o the average change in velocity is towards the centre o the circle. This is always the case and thus true or the instantaneous acceleration. For a mass m moving at a speed in uni orm circular motion o radius r,

Centripetal acceleration $a_{centripetal} = \frac{\;^2}{r}$ [in towards the centre o the circle]

A orce must have caused this acceleration. The value o the orce is worked out using Newton's second law:

Centripetal orce (CPF) $F_{centripetal} = m\, a_{centripetal}$

$= \frac{m\;^2}{r}$ [in towards the centre o the circle]

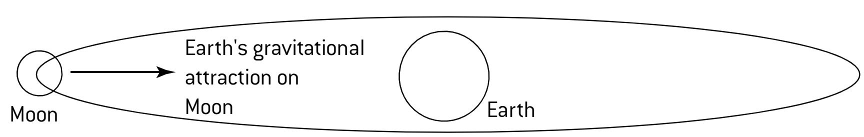
For example, i a car o mass 1500 kg is travelling at a constant speed o 20 m s^{-1} around a circular track o radius 50 m, the resultant orce that must be acting on it works out to be

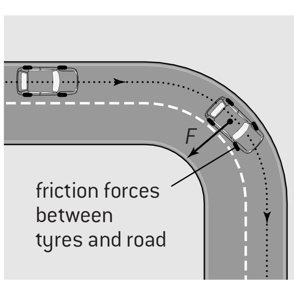
$$F = \frac{1500(20)^2}{50} = 12\,000 \text{ N}$$

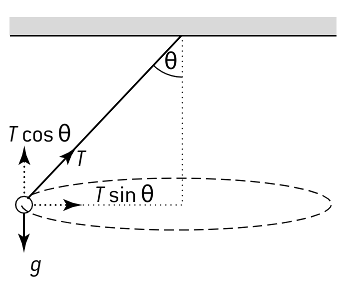
It is really important to understand that centripetal orce is NOT a new orce that starts acting on something when it goes in a circle. It is a way o working out what the total orce must have been. This total orce must result rom all the other orces on the object. See the examples below or more details.

One final point to note is that the centripetal orce does NOT do any work. (Work done = orce × distance **in the direction of the force**.)

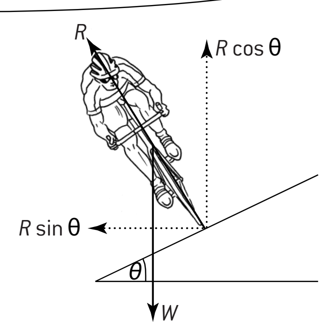
EXAMPLES







A conical pendulum – centripetal force provided by horizontal component of tension.

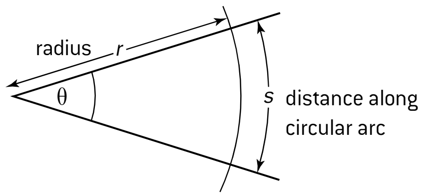


At a particular speed, the horizontal component of the normal reaction can provide all the centripetal force (without needing friction).

Angular velocit and vertical circular motion

RADIANS

Angles measure the raction o a complete circle that has been achieved. They can, o course, be measured in degrees (symbol: °) but in studying circular motion, the radian (symbol: rad) is a more use ul measure.



The raction o the circle that has been achieved is the ratio o arc length s to the circum erence:

$$\text{raction o circle} = \frac{s}{2\pi r}$$

In degrees, the whole circle is divided up into 360° which defines the angle θ as:

$$\theta(\text{in degrees}) = \frac{s}{2\pi r} \times 360$$

In radians, the whole circle is divided up into 2π radians which defines the angle θ as:

$$\theta(\text{in radians}) = \frac{s}{2\pi r} \times 2\pi = \frac{s}{r}$$

For small angles (less than about 0.1 rad or 5°), the arc and the two radii orm a shape that approximates to a triangle. Since radians are just a ratio, the ollowing relationship applies i working in radians:

$$\sin\theta \approx \tan\theta \approx \theta$$

Angle/°	Angle/radian
0	0.00
5	0.09
45	$0.74 = \frac{\pi}{4}$
60	1.05
90	$1.57 = \frac{\pi}{2}$
180	$3.14 = \pi$
270	$4.71 = \frac{3\pi}{2}$
360	$6.28 = 2\pi$

ANGULAR VELOCITY, ω, AND TIME PERIOD, T

An object travelling in circular motion must be constantly changing direction. As a result its velocity is constantly changing even i its speed is constant (uni orm circular motion). We define the average **angular velocity**, symbol (omega) as:

$$_{\text{average}} = \frac{\text{angle turned}}{\text{time taken}} = \frac{\Delta\theta}{\Delta t}$$

The units o angular velocity are radians per second (rad s^{-1}).

The instantaneous angular velocity is the rate o change o angle:

$$= \text{rate o change o angle} = \frac{d\theta}{dt}$$

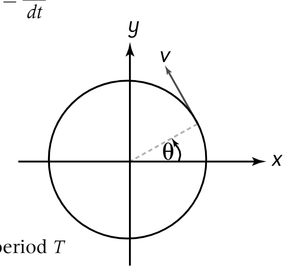
1. Link between and v

In a time Δt, the object rotates an angle $\Delta\theta$

$$\theta = \frac{s}{r} \;\therefore s = r\Delta\theta$$

$$v = \frac{s}{\Delta t} = \frac{r\Delta\theta}{\Delta t} = r$$

$$v = r$$



2. Link between and time period T

The time period T is the time taken to complete one ull circle. In this time, the total angle turned is 2π radians, so:

$$= \frac{2\pi}{T} \text{ or } T = \frac{2\pi}{}$$

3. Circular motion equations

Substitution o the above equations into the ormulae or centripetal orce and centripetal acceleration (page 65) provide versions that are sometime more use ul:

$$\text{centripetal acceleration, a} = \frac{v^2}{r} = r\ ^2 = \frac{4\pi^2 r}{T^2}$$

$$\text{centripetal orce, } F = \frac{mv^2}{r} = mr\ ^2 = \frac{4\pi^2\, mr}{T^2}$$

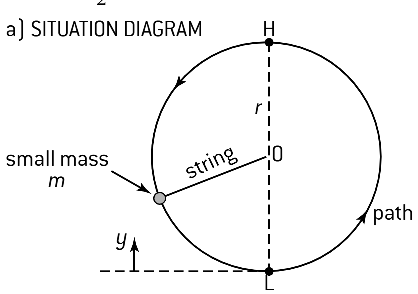
CIRCULAR MOTION IN A VERTICAL PLANE

Uni orm circular motion o a mass on the end o a string in a horizontal plane requires a constant centripetal orce to act and the magnitude o the tension in the string will not change. Circular motion in the vertical plane is more complicated as the weight o the object always acts in the same vertical direction. The object will speed up and slow down during its motion due to the component o its weight that acts along the tangent to the circle. The maximum speed will be when the object is at the bottom and the minimum speed will occur at the top. The tension in the string will also change during one revolution.

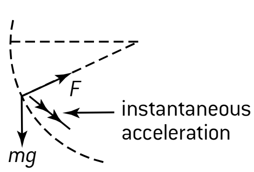
In a vertical circle, the tension o the string will always act at 90° to the object's velocity so this orce does no work in speeding it up or slowing it down. The conservation o energy means that:

$$mgy + \frac{1}{2}mv^2 = \text{constant}$$

a) SITUATION DIAGRAM



b) FREE-BODY DIAGRAM



1. At the top o the circle:,

The tension in the string, T, and the weight, mg, are in the same direction and add together to provide the CPF:

$$T_{\text{top}} + mg = \frac{mv_{\text{top}}^{\ 2}}{r}$$

To remain in the vertical circle, the object must be moving with a certain minimum speed. At this minimum top speed, $v_{\text{top min,}}$ the tension is zero and the centripetal orce is provided by the object's weight:

$$mg = \frac{m(v_{\text{top min}})^2}{r}$$

$$\text{v}_{\text{top min}} = \sqrt{rg}$$

2. At the bottom o the circle:,

The tension in the string, T, and the weight, mg, are in opposite directions and the resultant orce provides the CPF:

$$T_{\text{bottom}} - mg = \frac{mv_{\text{bottom}}^{\ 2}}{r}$$

In order to complete the vertical circle, the KE at the bottom o the circle must be large enough or the object to arrive at the top o the circle with su ficient speed($v_{\text{top min}} = \sqrt{rg}$) to complete the circle. Energetically the object gains PE ($= mg \times 2r$) so it must lose the same amount o KE:

$$\frac{1}{2}\, m(v_{\text{bottom min}})^2 - mg2r = \frac{1}{2}\, m(v_{\text{top min}})^2 = \frac{1}{2}\, mrg$$

$$\therefore (v_{\text{bottom min}})^2 - 4gr = rg$$

$$\therefore v_{\text{bottom min}} = \sqrt{5rg}$$

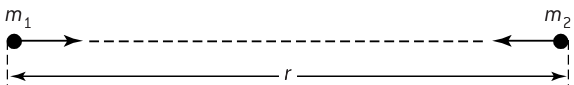
The mathematics in the above example (a mass on the end o a string) can also apply or any vehicle that is 'looping the loop'. In place o T, the tension in the rope, there is N, the normal reaction rom the sur ace.

Ne ton's la of gravitation

NEWTON'S LAW OF UNIVERSAL GRAVITATION

I you trip over, you will all down towards the ground.

Newton's theory o **universal gravitation** explains what is going on. It is called 'universal' gravitation because at the core o this theory is the statement that every mass in the Universe attracts all the other masses in the Universe. The value o the attraction between two **point** masses is given by an equation.



$$F \propto \frac{m_1 m_2}{r^2} \qquad F = \frac{G m_1 m_2}{r^2}$$

Universal gravitational constant $G = 6.67 \times 10^{-11}$ N m² kg⁻²

The ollowing points should be noticed:

- The law only deals with point masses.
- There is a orce acting on each o the masses. These orces are EQUAL and OPPOSITE (even i the masses are not equal).
- The orces are always attractive.
- Gravitation orces act between ALL objects in the Universe. The orces only become significant i one (or both) o the objects involved are massive, but they are there nonetheless.

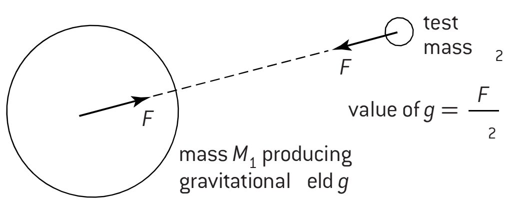
The interaction between two spherical masses turns out to be the same as i the masses were concentrated at the centres o the spheres.

GRAVITATIONAL FIELD STRENGTH

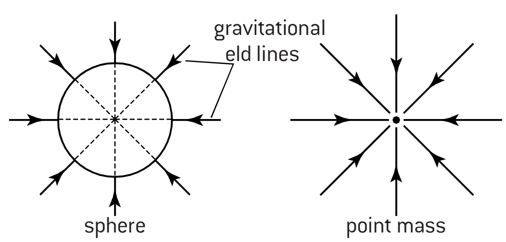
The table below should be compared with the one on page 61.

	Gravitational field strength
Symbol	g
Caused by...	Masses
A ects...	Masses
One type o ...	Mass
Simple orce rule:	All masses attract

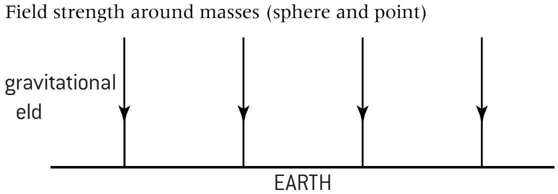
The gravitational field is there ore defined as the orce per unit mass. $g = \frac{F}{m}$ m = small point test mass



The SI units or g are N kg⁻¹. These are the same as m s⁻². Field strength is a vector quantity and can be represented by the use o field lines.



Field strength around masses (sphere and point)



Gravitational field near sur ace o the Earth

In the example on the le t the numerical value or the gravitational field can be calculated using Newton's law:

$$F = \frac{GMm}{r^2} \qquad g = \frac{GM}{r^2}$$

The gravitational field strength at the sur ace o a planet must be the same as the acceleration due to gravity on the sur ace.

Field strength is defined to be $\frac{\text{orce}}{\text{mass}}$

Acceleration = $\frac{\text{orce}}{\text{mass}}$ (rom $F = ma$)

For the Earth

$M = 6.0 \times 10^{24}$ kg

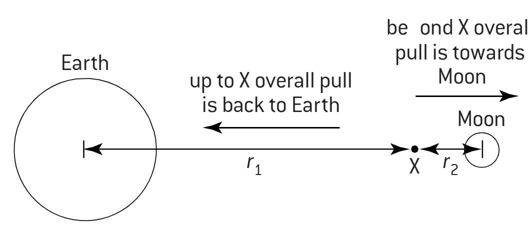
$r = 6.4 \times 10^{6}$ m

$$g = \frac{6.67 \times 10^{-11} \times 6.0 \times 10^{24}}{(6.4 \times 10^{6})^2} = 9.8 \text{ m s}^{-2}$$

EXAMPLE

In order to calculate the overall gravitational field strength at any point we must use vector addition. The overall gravitational field strength at any point between the Earth and the Moon must be a result o both pulls.

There will be a single point somewhere between the Earth and the Moon where the total gravitational field due to these two masses is zero. Up to this point the overall pull is back to the Earth, a ter this point the overall pull is towards the Moon.



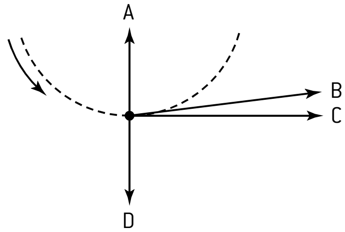
distance between Earth and Moon = $(r_1 + r_2)$

I resultant gravitational field at X = zero,

$$\frac{GM_{\text{Earth}}}{r_1^{\,2}} = \frac{GM_{\text{Moon}}}{r_2^{\,2}}$$

IB Quest ons – c rcular mot on and grav tat on

1. A ball is tied to a string and rotated at a uni orm speed in a vertical plane. The diagram shows the ball at its lowest position. Which arrow shows the direction o the net orce acting on the ball? [1]



2. A particle o mass m is moving with constant speed in uni orm circular motion. What is the total work done by the centripetal orce during one revolution? [1]

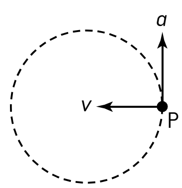
   A. Zero B. $\frac{m\,^{2}}{2}$

   C. $m\,^{2}$ D. $2\pi m\,^{2}$

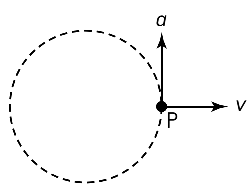
3. A particle P is moving anti-clockwise with constant speed in a horizontal circle.

   Which diagram correctly shows the direction o the velocity and acceleration a o the particle P in the position shown? [1]

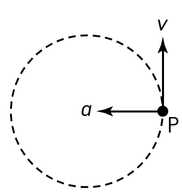
A.



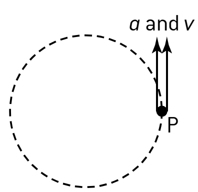
B.



C.

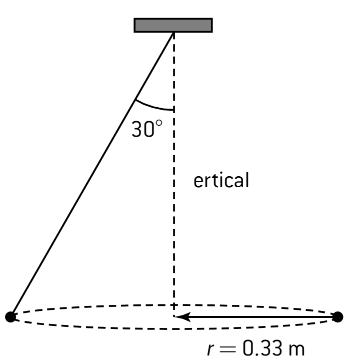


D.



4. This question is about circular motion.

   A ball o mass 0.25 kg is attached to a string and is made to rotate with constant speed along a horizontal circle o radius $r = 0.33$ m. The string is attached to the ceiling and makes an angle o 30° with the vertical.



   a) (i) On the diagram above, draw and label arrows to represent the orces on the ball in the position shown. [2]

   (ii) State and explain whether the ball is in equilibrium. [2]

   b) Determine the speed o rotation o the ball. [3]

5. This question is about gravitational fields.

   a) Define *gra itational field strength*. [2]

   b) The gravitational field strength at the sur ace o Jupiter is 25 N kg^{-1} and the radius o Jupiter is 7.1×10^{7} m.

   (i) Derive an expression or the gravitational field strength at the sur ace o a planet in terms o its mass M, its radius R and the gravitational constant G. [2]

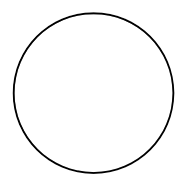
   (ii) Use your expression in (b)(i) above to estimate the mass o Jupiter. [2]

6. Gravitational fields and potential

   a) Derive an expression or the gravitational field strength as a unction o distance away rom a point mass M. [3]

   b) The radius o the Earth is 6400 km and the gravitational field strength at its sur ace is 9.8 N kg^{-1}. Calculate a value or the mass o the Earth. [2]

   c) On the diagram below draw lines to represent the gravitational field outside the Earth. [2]



   d) A satellite that orbits the Earth is in the gravitational field o the Earth. Discuss why an astronaut inside the satellite eels weightless. [3]