6 CIRCULAR MOTION AND GRAVITATION

Introduction

Two apparently distinct areas of physics are linked in this topic: motion in a circle and the basic ideas of gravitation. But of course they are not distinct at all. The motion of a satellite about its planet involves both a consideration of the gravitational force and the mechanics of motion in a circle. Man cannot travel beyond the Earth without a knowledge of both these aspects of Physics.

6.1 Circular motion

Understanding

- Period, frequency, angular displacement, and angular velocity
- Centripetal force
- Centripetal acceleration

Nature of science

The drive to develop ideas about circular motion came from observations of the universe. How was it that astronomical objects could move in circular or elliptical orbits? What kept them in place in their motion? Scientists were able to deduce that there must be a force acting radially inwards for every case of circular motion that is observed. Whether it is a bicycle going around a corner or a planet orbiting its star, the physics is the same.

Applications and skills

- Identifying the forces providing the centripetal forces such as tension, friction, gravitational, electrical, or magnetic
- Solving problems involving centripetal force, centripetal acceleration, period, frequency, angular displacement, linear speed, and angular velocity
- Qualitatively and quantitatively describing examples of circular motion including cases of vertical and horizontal circular motion

Equations

- speed–angular speed relationship: $v = \omega r$
- centripetal acceleration: $a = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2}$
- centripetal force: $F = \frac{mv^2}{r} = m\omega^2 r$

▲ Figure 1 A fairground carousel.

Moving in a circle

Most children take great delight in an object on a string whirling in a circle – though they may be less happy with the consequences when the string breaks and the object hits a window! Rides at a theme park and trains on a railway are yet more examples of movement in a circle. What is needed to keep something rotating at constant speed?

The choice of term (as usual in physics!) is very deliberate. In circular motion we say that the "speed is constant" but not the "velocity is constant".

Velocity, as a vector quantity, has both magnitude *and* direction. The object on the string has a constant speed but the direction in which the object is moving is changing all the time. The velocity has a constant *magnitude* but a changing *direction*. If either of the two parts that make up a vector change, then the vector is no longer constant. Whenever velocity changes (even if it is only the direction) then the object is accelerated.

Understanding the physics of this acceleration is the key to understanding circular motion. But before looking at how the acceleration arises we need a language to describe the motion.

Angular displacement

The angle moved around the circle by an object from where its circular motion starts is known as the **angular displacement**. Unlike the linear displacement used in Topic 2, angular displacement will not be considered to be a vector in IB Physics. Angular displacement is the angle through which the object moves and it can be measured in degrees (°) or in radians (rad). Radians are more commonly used than degrees in this branch of physics. If you have not met radians before, read about the differences between radians and degrees.

Nature of science

Radians or degrees

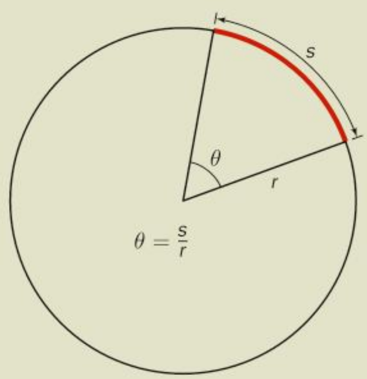
Calculations of circular motion involve the use of angles. In any science you studied before starting this course you will almost certainly have measured all angles in degrees.

1° (degree) is defined to be $\frac{1}{360}$ th of the way around a circle.

In some other areas of physics (including circular motion) there is an alternative measure of angle that is much more convenient, the **radian**. Radians are based on the geometry of the arc of a circle.

1 radian (abbreviated as rad) is defined as the angle equal to the circumference of an arc of a circle divided by the radius of the circle. In symbols

$$\theta = \frac{s}{r}$$



▲ Figure 2 Definition of radian.

Going around the circle once means travelling around the circumference; this is a distance of $2\pi r$. The angle θ in radians subtended by the whole circle is $\frac{2\pi r}{r} = 2\pi$ rad.

So $360° = 2\pi = 6.28$ rad

and 1 rad $= 57.3°$

Sometimes, the radian numbers are left as fractions, so

$$90^\circ = \frac{\pi}{2}\left(\frac{1}{4}\text{ round the circle}\right),$$

$$30^\circ = \frac{\pi}{6}\left(\frac{1}{12}\text{ round the circle}\right)$$

and so on.

To convert other values for yourself, use the equation $\frac{\text{angle in degree}}{360} = \frac{\text{angle in radians}}{2\pi}$

There are some similarities between the sine of an angle and the angle in radians. The two quantities are compared in this Nature of Science box which shows $\sin\theta$ and θ in radians. Notice that, as θ becomes smaller, $\sin(\theta)$ and θ become closer together. From angles of 10° down to 0, the differences between $\sin\theta$ and θ are very small and in some calculations and proofs we treat $\sin\theta$ and θ as being equal (this is known as "the small angle approximation"). For small angles $\cos\theta$ approximates to zero radians.

To illustrate this, here are the values of $\sin\theta$ and θ in radians for four angles: 90°, 45°, 10°, and 5°. Notice how similar the sine values and the radians are for 10° and 5°.

$\sin(90^\circ) = 1.000$; $\frac{\pi}{2}\text{ rad} = 1.571\text{ rad}$

$\sin(45^\circ) = 0.707$; $\frac{\pi}{4}\text{ rad} = 0.785\text{ rad}$

$\sin(10^\circ) = 0.175$; $\frac{\pi}{18}\text{ rad} = 0.174\text{ rad}$

$\sin(5^\circ) = 0.087$; $\frac{\pi}{36}\text{ rad} = 0.087\text{ rad}$

Finally, a practical point: Scientific and graphic calculators work happily in either degrees or radians (and sometimes in another type of angular measure known as "grad" too). But the calculator has to be "told" what to expect! Always check that your calculator is set to work in radians if that is what you want, or in degrees if those are the units you are using. You will lose calculation marks in an examination if you confuse the calculator!

Angular speed

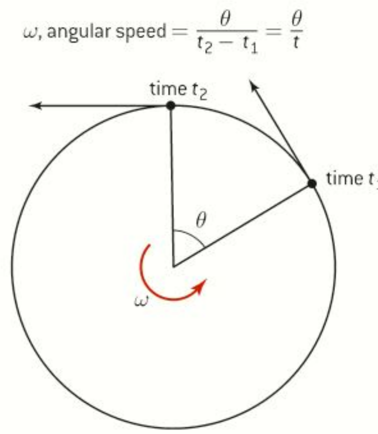
In Topic 2 we used the term *speed* to mean "linear speed". When the motion is in a circle there is an alternative: **angular speed**, this is given the symbol ω (the lower-case Greek letter, omega).

$$\textit{average angular speed} = \frac{\textit{angular displacement}}{\textit{time for the angular displacement to take place}}$$

Figure 3 shows how things are defined and you will see that in symbols the definition becomes

$$\omega = \frac{\theta}{t}$$

where θ is the angular displacement and t is the time taken for the angular displacement.



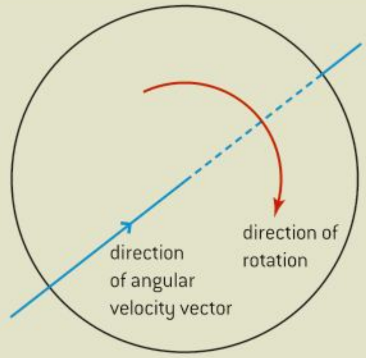
▲ Figure 3 Angular speed.

Nature of science

Angular speed or angular velocity?

You may be wondering about the distinction between angular speed and angular velocity, and whether angular velocity is a vector similar to linear velocity.

The answer is that angular velocity is a vector but an unusual one. It has a magnitude equal to the angular speed, but its direction is surprising! The direction is along the axis of rotation, in other



▲ Figure 4 Angular velocity direction.

words, through the centre of the circle around which the object is moving and perpendicular to the plane of the rotation.

The direction follows a clockwise corkscrew rule so that in this example the direction of the angular velocity vector is into the plane of the paper.

In the IB course, only the angular speed – the scalar quantity – is used.

Period and frequency

The time taken for the object to go round the circle once is known as the **periodic time** or simply the **period** of the motion, it has the symbol T. In one period, the angular distance travelled is 2π rad. So,

$$T = \frac{2\pi}{\omega}$$

When T is in seconds the units of ω are radians per second, abbreviated to rad s^{-1}.

If you have already studied waves in this course, you might have met the idea of **time period** – the time for one cycle. Another quantity that is associated with T is **frequency**. Frequency is the number of times an object goes round a circle in unit time (usually taken to be 1 second), so one way to express the unit of frequency would be in "per second" or s^{-1}. However, the unit of frequency is re-named after the 19th century physicist Heinrich Hertz and is abbreviated to Hz. There is a link between T and f so that:

$$T = \frac{1}{f}$$

This leads to a link between ω and f

$$\omega = 2\pi f$$

Worked example

A large clock on a building has a minute hand that is 4.2 m long.

Calculate:

a) the angular speed of the minute hand

b) the angular displacement, in radians, in the time periods

(i) 12 noon to 12.20

(ii) 12 noon to 14.30.

c) the linear speed of the tip of the minute hand.

Solution

a) The minute hand goes round once (2π rad) every hour.

One hour is 3600 s

$$\text{angular speed} = \frac{\textit{angular displacement}}{\textit{time taken}} = \frac{2\pi}{3600} = 0.001\,75 \text{ rad s}^{-1}$$

b) (i) 20 minutes is $\frac{1}{3}$ of 2π, so $\frac{2\pi}{3}$ rad

(ii) 2.5 h is $2\pi \times 2.5 = 5\pi$ rad

c) $v = r\omega = 4.2 \times 0.001\,75 = 0.007\,33 \text{ m s}^{-1} = 7.3 \text{ mm s}^{-1}$

Linking angular and linear speeds

Sometimes we know the linear speed and need the angular speed or vice versa.

The link is straightforward: When the circle has a radius r the circumference is $2\pi r$, and T, is the time taken to go around once. So the linear speed of the object along the edge of the circle v is

$$v = \frac{2\pi r}{T}$$

Rearranging the equation gives

$$T = \frac{2\pi r}{v}$$

We have just seen that

$$T = \frac{2\pi}{\omega}$$

so equating the two equations for T gives

$$\frac{2\pi r}{v} = \frac{2\pi}{\omega}$$

Cancelling the 2π and rearranging gives

$$v = \omega r$$

Notice that, in both this equation and in the earlier equation $s = \theta r$, the radius r multiplies the angular term to obtain the linear term. This is a consequence of the definition of the angular measure.

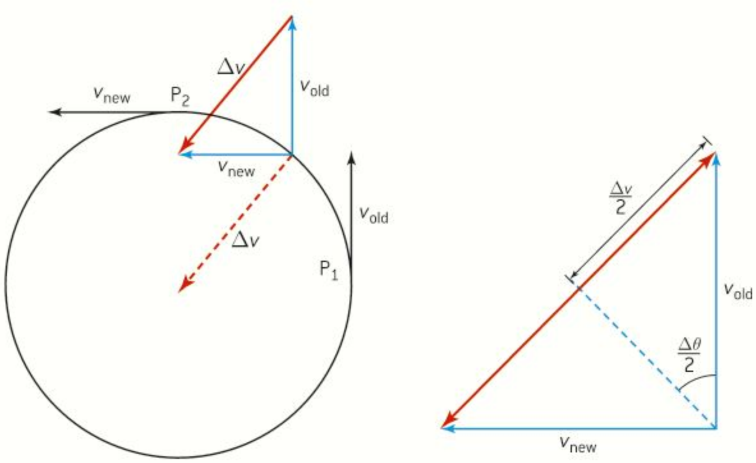
Centripetal acceleration

Earlier we showed that an object moving at a constant angular speed in a circle is being accelerated. Newton's first law tells us that, for any object in which the direction of motion or the speed is changing, there must be an external force acting. In circular motion the direction is constantly changing and so the object accelerates and there must be a force acting on it to cause this to happen. In which direction do the force and the acceleration act, and what are their sizes?

The diagram shows two points P_1 and P_2 on the circle together with the velocity vectors v_{old} and v_{new} at these points. The vectors are the same length as each other because the speed is constant. However, v_{old} and v_{new} point in different directions because the object has moved round the circle by an angular distance $\Delta\theta$ between P_1 and P_2. Acceleration is, as usual,

$$\frac{\text{change of velocity}}{\text{time taken for the change}}$$

The change in velocity is the change-of-velocity vector Δv that has to be added to v_{old} in order to make it become the same length and direction as v_{new}. Identify these vectors on the diagram.



▲ Figure 5 Proof of centripetal acceleration direction.

Notice that v_{old} and v_{new} slide round the circle to meet. Where does the new vector Δv point? The answer is: to the centre of the circle. This is an "averaging" process to find out what the difference is between v_{old} and v_{new} half-way between the two points.

This averaging can be taken further. The time, Δt, to go between P_1 and P_2, and the linear distance around the circle between P_1 and P_2 (which is $r\theta$) are related by

$$\Delta t = \frac{r\Delta\theta}{v}$$

Using some trigonometry on the diagram shows that

$$\frac{\Delta v}{2} = v\sin\left(\frac{\Delta\theta}{2}\right)$$

The size of the average acceleration $\boldsymbol{a}$ that is directed towards the centre of the circle is

$$\boldsymbol{a} = \frac{\Delta v}{\Delta t} = \frac{2v\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{2r}{v}\frac{\Delta\theta}{2}}$$

This can be written as

$$\boldsymbol{a} = \frac{v^2}{r}\,\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}$$

When $\Delta\theta$ is very small, the ratio $\frac{\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}$ is almost exactly equal to 1 and so the instantaneous acceleration a when P_1 and P_2 are very close together is

$a = \frac{v^2}{r} = \omega^2 r = v\omega$ directed to the centre of the circle.

This acceleration is at 90° to the velocity vector and it points inwards to the centre of the circle.

The force that acts to keep the object moving in a circle is called the **centripetal force** and this force leads to a **centripetal acceleration**. (The origin of the word centripetal comes from two Latin words *centrum* and *petere* – literally "to lead to the centre".)

Nature of science

Linking it together

Notice that some of these equations have interesting links elsewhere: $mv\omega$ is, for example, the magnitude of the linear momentum multiplied by ω. Try to be alert for these links as they will help you to piece your physics together.

Centripetal force

Newton's second law of motion in its simpler form tells us that $F = ma$ using the usual symbols.

The second law applies to the force that provides the centripetal acceleration, so the magnitude of the force $= m\frac{v^2}{r} = m\omega^2 r = mv\omega$. The question we need to ask for each situation is: what force provides the centripetal force for that situation? The direction of this force must be along the radial line between the object and the centre of the circle.

Investigate!

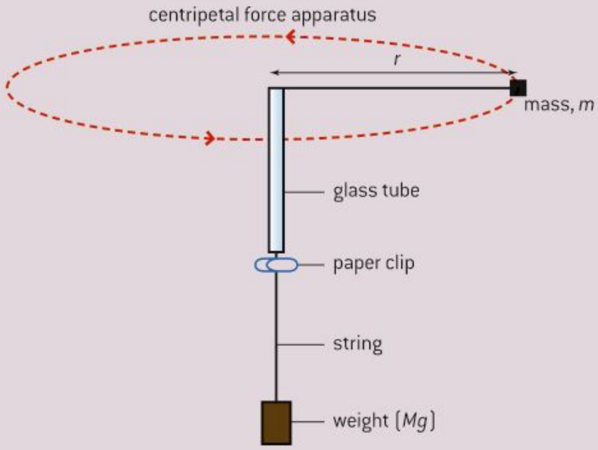
Investigating how F varies with m, v and r

This experiment tests the relationship

$$m\frac{v^2}{r} = Mg$$

- To do this a bung is whirled in a horizontal circle with a weight hanging from one end of a string and mass (rubber bung) on the other end.

- A paper clip is attached to the string below a glass tube. The clip is used to ensure that the radius of rotation of the bung is constant – the bung should be rotated at a speed so that the paper clip just stays below the glass tube.



▲ Figure 6 Centripetal force, mass, and speed.

- The tension in the string is the same everywhere (whether below the glass tube or above in the horizontal part). This tension is F in the equation and is equal to Mg where M is the mass of the weight (hanging vertically).
- Use a speed at which you can count the number of rotations of m in a particular time and from this work out the linear speed v of mass m.

To verify the equation you need to test each variable against the others. There are a number of possible experiments in each of which one variable is held constant (a control variable), one is varied (the independent variable), and the third (the dependent variable) is measured. One example is:

Variation of v with r

- In this experiment, m and M must be unchanged. Move the clip to change r, and for each value of r, measure v using the method given above.
- Analysis:

$$\frac{v^2}{r} = constant$$

- A graph of v^2 against r ought to be a straight line passing through the origin. Alternatively you could, for each experimental run, simply divide v^2 by r and look critically at the answer (which should be the same each time) to see if the value is really constant. If going down this route, you ought to assess the errors in the experiment and put error limits on your $\frac{v^2}{r}$ value.
- What are the other possible experimental tests?
- In practice the string cannot rotate in the horizontal plane because of its own weight. How can you improve the experiment or the analysis to allow for this?

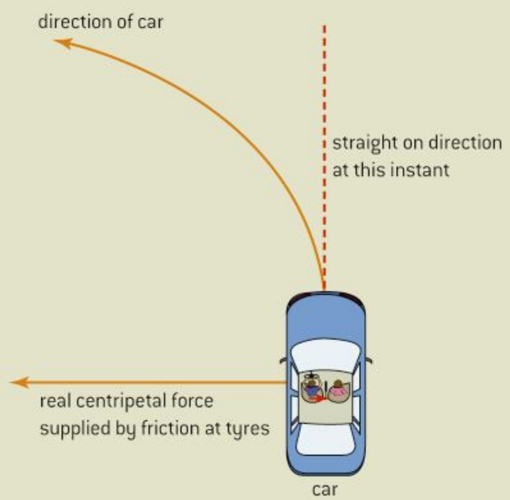
Nature of science

Centripetal or centrifugal?

When discussing circular motion, you will almost certainly have heard the term “centrifugal force” – probably everywhere except in a physics laboratory! In this course we have spoken exclusively about “centripetal force”. Why are there two terms in use?

It should now be clear to you how circular motion arises: a force acts to the centre of the circle around which the object is moving. The alternative idea of centrifugal force comes from common experience. Imagine you are in a car going round a circle at high speed. You will undoubtedly feel as if you are being “flung outwards”.

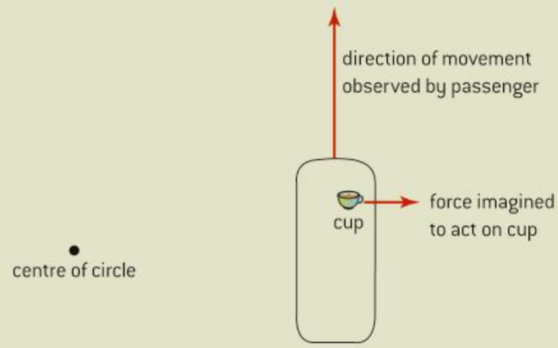
One way to explain this is to imagine the situation from the vantage point of a helicopter hovering



▲ Figure 7 Centripetal forces in a car seen from above.

stationary above the circle around which the car is moving. From the helicopter you will see the passenger attempting to go in a straight line (Newton's first law), but the passenger is forced to move in a circle through friction forces between passenger and seat. If the passenger were sitting on a friction-less seat and not wearing a seat belt, then he or she will not get the "message" that the car is turning. The passenger continues to move in a straight line eventually meeting the door that is turning with the car. If there were no door, what direction will the passenger take?

Another way to explain this is to imagine yourself in the car as it rotates. This is a rotating frame of reference that is accelerating and as such cannot obey Newton's laws of motion. You instinctively think that the rotating frame is actually stationary. Therefore your tendency to go in what you believe to be a straight line actually feels like an outward force away from the centre of the circle (remember the rest of the world now rotates round you, and your straight line is actually part of a circle). Think about a cup of coffee sitting on the floor of the car. If there is insufficient friction at the base of the cup, the cup will slide to the side of the car. In the inertial frame of reference (the Earth) the cup is trying to go in a straight line. In your rotating frame of reference you have to "invent" a force acting outwards from the centre of the circle to explain the motion of the cup.

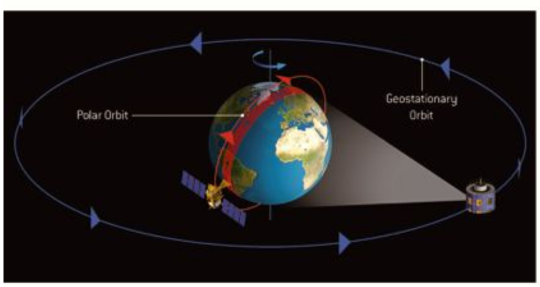


▲ Figure 8 Rotation forces.

There are many examples of changing a reference frame in physics: research the Foucault pendulum and perhaps go to see one of these fascinating pendulums in action. Look up what is meant by the Coriolis force and find out how it affects the motion of weather systems in the northern and southern hemispheres.

One of the tricks that physicists often use is to change reference frames – it's all part of the nature of science to adopt alternative frames of reference to make explanations and theories more accessible.

One last tip: Don't use explanations based on centrifugal force in an IB examination. The real force is centripetal; centrifugal force was invented to satisfy Newton's second law in an accelerated frame of reference.



▲ Figure 9 Satellites in orbit.

Centripetal accelerations and forces in action

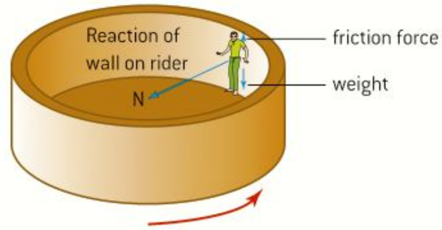
Satellites in orbit

Figure 9 shows satellites in a circular orbit around the Earth. Why do they follow these paths? Gravitational forces act between the centre of mass of the Earth and the centre of mass of the satellite. The direction of the force acting on the satellite is always towards the centre of the planet and it is the gravity that supplies the centripetal force.

Amusement park rides

Many amusement park rides take their passengers in curved paths that are all or part of a circle. How does circular motion provide a thrill?

In the type of ride shown in figure 10, the people are inside a drum that rotates about a vertical axis. When the rotation speed is large enough the people are forced to the sides of the drum and the floor drops away. The people are quite safe however because they are "held" against the inside of the drum as the reaction at the wall provides the centripetal force to keep them moving in the circle. The people in the ride feel the reaction between their spine and the wall. Friction between the rider and the wall prevents the rider from slipping down the wall.



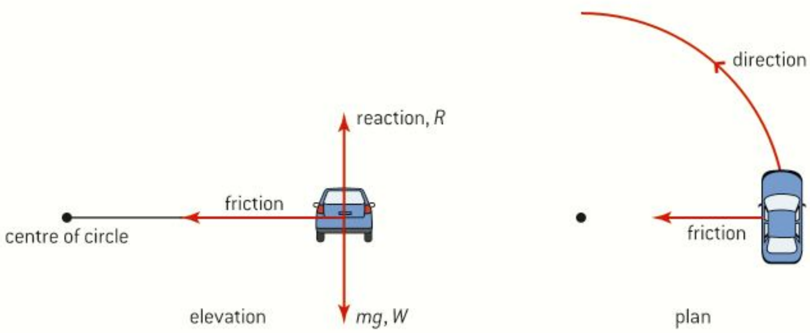
▲ Figure 10 The rotor in action.

Turning and banking

When a driver wants to make a car turn a corner, a resultant force must act towards the centre of the circle to provide a centripetal force. The car is in vertical equilibrium (the driving surface is horizontal) but not in horizontal equilibrium.

Turning on a horizontal road

For a horizontal road surface, the friction acting between the tyres and the road becomes the centripetal force. The friction force is related to the coefficient of friction and the normal reaction at the surface where friction occurs.



▲ Figure 11 Car moving in a circle.

If the car is not to skid, the centripetal force required has to be less than the frictional force

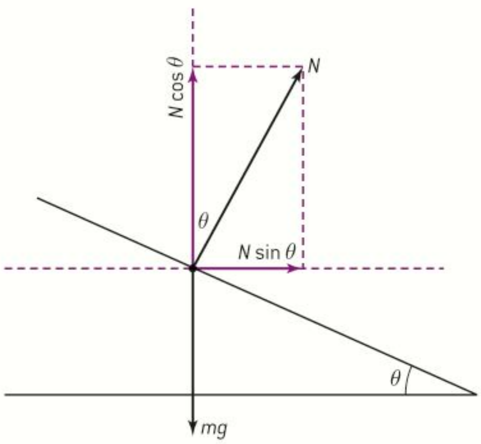
$$m\frac{v^2}{r} < \mu_s mg$$

where μ_s is the static coefficient of friction. Note that when the vehicle is already skidding the "less than" sign becomes an equality and the dynamic coefficient of friction should be used.

This rearranges to give a maximum speed of $v_{max} = \sqrt{\mu_d rg}$ for a circle of radius r.

Banking

Tracks for motor or cycle racing, and even ordinary roads for cars are sometimes **banked** (figures 12 and 13). The curve of the banked road surface is inclined at an angle so that the normal reaction force contributes to the centripetal force that is needed for the vehicle to go round the track



▲ Figure 12 Forces in banking.

at a particular speed. Bicycles and motorcycles can achieve the same effect on a level road surface by "leaning in" to the curve. Tyres do not need to provide so much friction on a banked track compared to a horizontal road; this reduces the risk of skidding and increases safety.

Although you will not be asked to solve mathematical problems on this topic in your IB Physics examination, you do need to understand the principles that underpin banking.

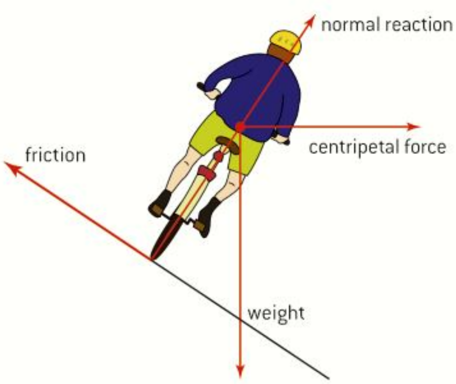
Figure 12 shows forces acting on a small sphere rolling round a track. This is simplified to a point object moving in a circle to remove the complications of two or four wheels. A horizontal centripetal force directed towards the centre of the circle is needed for the rotation. The other forces that act on the ball are the force normal to the surface (which is at the banking angle θ) and its weight acting vertically down. The vector sum of the horizontal components of the weight and the normal force must equal the centripetal force.

Looking at this another way, if N is the normal force then the centripetal force is equal to

$$N \sin\theta$$

The normal force resolved vertically is $N\cos\theta$ and is, of course, equal and opposite to mg. So $F_{\text{centripetal}} = \left(\frac{mg}{\cos\theta}\right)\ \sin\theta = mg\tan\theta$

$$F_{\text{centripetal}} = \frac{mv^2}{r} \text{ and therefore } \tan\theta = \frac{v^2}{gr}$$



The banking angle is correct at a particular speed and radius. Notice that it does not depend on the mass of the vehicle so a banked road works for a cyclist and a car, provided that they are going at the same speed.

Some more examples of banking:

- Commercial airline pilots fly around a banked curve to change the direction of a passenger jet. If the angle is correct, the passengers will not feel the turn, simply a marginal increase in weight pressing down on their seat).
- Some high-speed trains tilt as they go around curves so that the passengers feel more comfortable.

▲ Figure 13 Cycle velodrome.

Moving in a vertical circle

So far the examples have been of motion around a horizontal circle. People will queue for a long time to experience moderate fear on a fairground attraction like the rollercoaster in figure 14. The amount of thrill from the ride depends on its height, speed, and also the forces that act on the riders.

How is the horizontal situation modified when the circular motion of the mass is in a vertical plane?

1 What are the forces acting when the motion is in a vertical circle?

Imagine a mass on the end of a string that is moving in a vertical circle at constant speed.

Look carefully at figure 15 and notice the way the tension in the string changes as the mass goes around.

▲ Figure 14 Theme park ride.

Begin with the case when the string is horizontal, at point A. The weight acts downwards and the tension in the string is the horizontal centripetal force towards the centre of the circle.

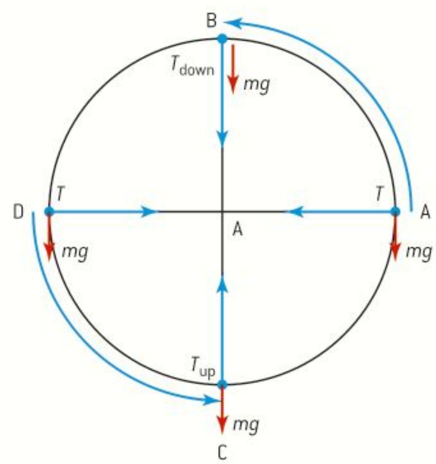
The mass continues to move upwards and reaches the top of the circle at B. At this point the tension in the string and the weight both act downwards. Thus:

$$T_{\text{down}} + mg = m\frac{v^2}{r}$$

and therefore

$$T_{\text{down}} = m\frac{v^2}{r} - mg$$

The weight of the mass combines with the tension to provide the centripetal force and so the tension required is less than the tension T when the string is horizontal.



▲ Figure 15 Forces in circular motion in a vertical plane.

At C, the bottom of the circle, the tension and the weight both act vertically but in opposite directions and so

$$T_{\text{up}} = m\frac{v^2}{r} + mg$$

At the bottom, the string tension must overcome weight and also provide the required centripetal force.

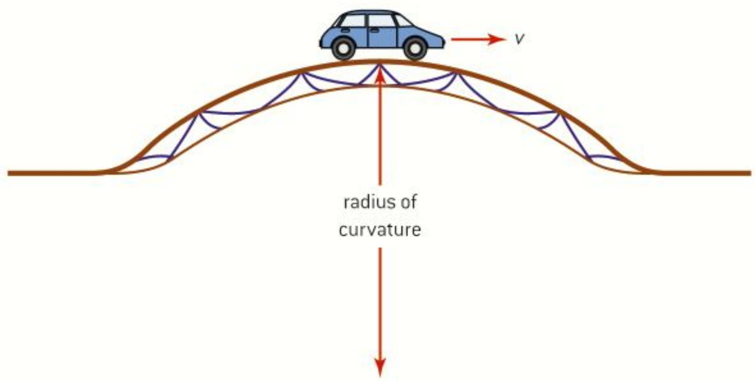
As the mass moves around the circle, the tension in the string varies continuously. It has a minimum value at the top of the circle and a maximum at the bottom. The bottom of the circle is the point where the string is most likely to break. If the maximum breaking tension of the string is T_{break}, then, for the string to remain intact,

$$T_{\text{break}} > m\frac{v^2}{r} + mg$$

and the linear speed at the bottom of the circle must be less than

$$\sqrt{\frac{r}{m}(T_{\text{break}} - mg)}$$

If this seems to you to be a very theoretical idea without much practical value, think about a car going over a bridge. If you assume that the shape of the bridge is part of a circle, then there is a radius of curvature r. What is the speed at which the car will lose contact with the bridge?



▲ Figure 16 Car going over a bridge.

This is the case considered above, where the object, in this case the car, is at the top of the circle. What is the "tension" (in this case the force between car and road) if the car wheels are to lose contact with the bridge? To answer this question, you might begin with a free-body diagram. You should be able to show that the car loses contact at a speed equal to $\sqrt{gr}$.

2 How does speed change when motion is in a vertical circle?

Not all circular motion in a vertical circle is at a constant speed. As a mass moves upwards it slows as kinetic energy is transferred to gravitational potential energy (if there is nothing to keep it moving at constant speed). At the top of the motion the mass must not stop moving or even go too slowly, because if it did then the string would lose its tension. The motion would no longer be in a circle.

The centripetal force F_c needed to maintain the motion is $F_c = m\frac{v^2}{r}$ as usual, at the top of the circle, if F_c is supplied entirely by gravity then

$$F_c = mg = m\frac{v^2}{r}$$

Just for an instant, the object is in free-fall.

The equation can be rearranged to give

$$v_{top} = \sqrt{gr}$$

and this is the minimum speed at the top of the circle for which the motion will still be circular. The minimum speed does not depend on mass.

Energy is conserved assuming that there are no losses (for example, to internal energy as a result of air resistance as the mass goes round). Equating the energies:

kinetic energy at top + gravitational potential energy difference between top and bottom = kinetic energy at the bottom

and

$$\frac{1}{2}mv_{top}{}^2 + mg(2r) = \frac{1}{2}mv_{bottom}{}^2$$

By substituting for both tensions, T_{bottom} and T_{top}, it is possible to show that

$$T_{bottom} = T_{top} + 6m$$

You can find this proof on the website.

Worked examples

1 A hammer thrower in an athletics competition swings the hammer on its chain round 7.5 times in 5.2 s before releasing it. The hammer describes a circle of radius 4.2 m and has a mass of 4.0 kg. Assume that the hammer is swung in a horizontal circle and that the chain is horizontal.

a) Calculate, for the rotation:

(i) the average angular speed of the hammer

(ii) the average tension in the chain.

b) Comment on the assumptions made in this question.

Solution

a) (i) 7.5 revolutions $= 15\pi$ rad

angular speed $= \frac{15\pi}{5.2} = 9.1$ rad s^{-1}

(ii) Tension in the chain = centripetal force required for rotation centripetal force $= mr\omega^2 = 4.0 \times 4.2 \times 9.1^2 = 1400$ N

b) The thrower usually inclines the plane of the circle at about 45° to the horizontal in order to achieve maximum range. Even if the plane were horizontal, then the weight of the hammer would contribute to the system so that a component of the tension in the chain must allow for this. Both assumptions are unlikely.

6.2 Newton's law of gravitation

Understanding

- Newton's law of gravitation
- Gravitational field strength

Nature of science

Newton's insights into mechanics and gravitation led him to develop laws of motion and a law of gravitation. One of his motion laws and the law of gravitation are mathematical in nature, two of the motion laws are descriptive. None of these laws can be proved and there is no attempt in them to explain why the masses are accelerated under the influence of a force, or why two masses are attracted by the force of gravity. Newton's ideas about motion have been subsequently modified by the work of Einstein. The questioning and insight that leads to the development of laws are fundamental to the nature of science.

Applications and skills

- Describing the relationship between gravitational force and centripetal force
- Applying Newton's law of gravitation to the motion of an object in circular orbit around a point mass
- Solving problems involving gravitational force, gravitational field strength, orbital speed, and orbital period
- Determining the resultant gravitational field strength due to two bodies

Equations

- Newton's law of gravitation: $F = G\frac{Mm}{r^2}$
- gravitational field strength: $g = \frac{F}{m}$
- gravitational field strength and the gravitational constant: $g = G\frac{M}{r^2}$

Nature of science

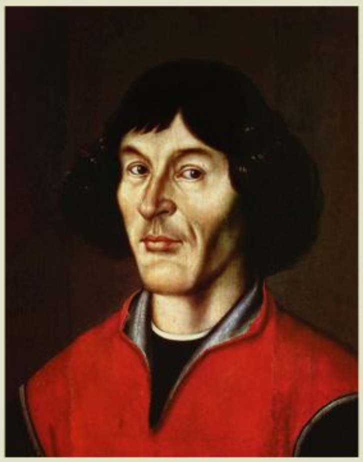
Scientists from the past

Three of the great names from the history of astronomy and physics were Copernicus, Tycho Brahe, and Kepler. Their contributions were linked and ultimately led to the important gravitational work of Newton. Try to find out something about these astronomers. During the lifetimes of these scientists, science was carried out in a very different way from today.

The realization that the Earth orbits the Sun rather than the Sun orbiting the Earth was one of the great developments in scientific understanding.

Galileo Galilei and other scientists of the 16th century overcame cultural, philosophical and religious prejudices, and some even suffered persecution for the scientific truths they had discovered. As we study the work of these pioneers we should remember that scientists in past times were not always as free as those today.

Research the life of Galileo (we often drop the second name) and explore why he and others came into conflict with the Roman Catholic Church over their scientific beliefs.



Copernicus (Mikolaj Kopernik)

Tycho Brahe

Johannes Kepler

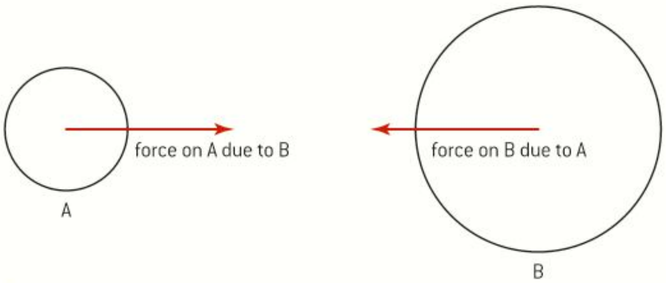
Isaac Newton

Galileo Galilei

▲ Figure 1 An astronomers' portrait gallery.

Gravitational field strength

Like electrostatics, gravity acts at a distance and is an example of a force that has an associated force field. Imagine two masses in deep space with no other masses close enough to influence them. One mass (call it A) is in the force field due to the second mass (B) and a force acts on A. B is in the gravitational field of A and also experiences a force. These two forces have an equal magnitude (even though the masses may be different) but act in opposite directions.

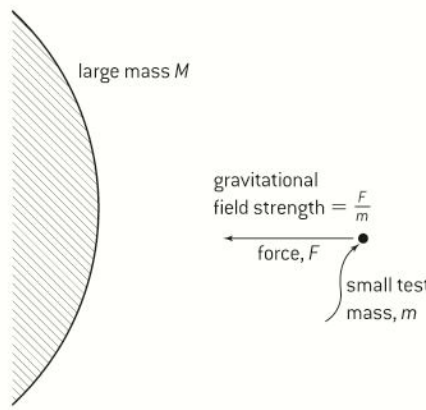


▲ Figure 2 Gravitational forces between two masses.

If both masses are small, the size of a human, say, the force of gravity is extremely weak. Only when one of the masses is as large as a planet does the force become noticeable to us. However, whatever the size of the mass we need a way to measure the strength of the field to which it gives rise. Gravitational forces are the weakest of all the forces in nature and so require large amounts of mass for the force to be felt.

The strength of a gravitational field is defined using the idea of a **small test mass**. This test mass has to be so small that it does not disturb the field being measured. If the test mass is large then it will exert a force of its own on the mass that produces the field being measured. The test mass would then accelerate the other mass and alter the arrangement that is being measured.

The gravitational force that acts on the small test mass has both magnitude and direction. These are shown in the diagram. The test mass will accelerate in this direction if it is free to move.



▲ Figure 3 Definition of gravitational field strength.

Defining gravitational field strength

The concept of the small test mass leads to a definition of the strength of the gravitational field.

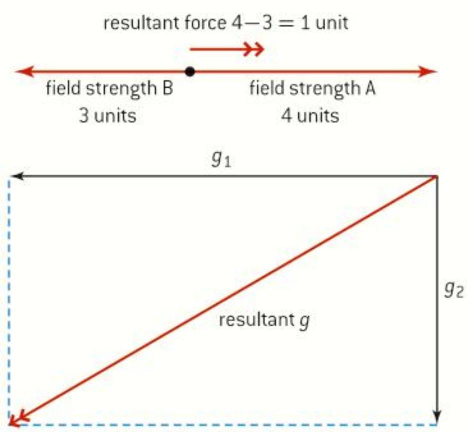
If the mass of the test mass is m, and the field is producing a gravitational force of F on the test mass, then the **gravitational field strength** (given the symbol g) is defined as

$$g = \frac{F}{m}$$

The units of gravitational field strength are N kg^{-1}.

Since F is a vector and m a scalar, it follows that g is a vector quantity and that its direction is that of F.

A formal definition in words is that **gravitational field strength at a point is the force per unit mass experienced by a small point mass placed at that point.** This definition requires that the test mass is not just small but is also an (infinitesimally) small point in space. You might want to consider what the effect might be if the test mass has a shape and extends in space.



▲ Figure 4 Adding field strengths vectorially.

So far the discussion has been limited to the gravitational field produced by one point mass. How does the situation change if there is more than one mass, excluding the test mass itself?

Field strength is independent of the magnitude of the point test mass (we divided F by m to achieve this). So the vector field strengths can be added together (figure 4).

In IB Physics examinations you will only be asked to add the field strengths of masses that all lie on the same straight line. But even in two dimensions, the addition is straightforward using the ideas of vector addition by drawing or by calculation.

g and the acceleration due to gravity

Sometimes students are surprised that the symbol g is used for the gravitational field strength, they think there might be a risk of confusion with g the acceleration due to gravity! However this does not happen.

At the Earth's surface (using Newton's second law)

acceleration due to gravity at the surface

$$= \frac{\text{force on a mass at the surface due to gravity}}{\text{size of the mass}}$$

So the acceleration $= \frac{F}{m}$ but $\frac{F}{m}$ is also the definition of gravitational field strength so the acceleration due to gravity = gravitational field strength = g.

The magnitude of the gravitational field strength (measured in N kg^{-1}) is equal to the value of the acceleration due to gravity (measured in m s^{-2}). You should be able to show that N $kg^{-1} \equiv$ m s^{-2}. (The symbol $\equiv$ means "is equivalent to".)

Newton's law of gravitation – an inverse-square law

Isaac Newton (1642–1727) was a British scientist who consolidated the work of others and who added important insights of his own. During his life he contributed to the study of optics, mechanics and gravitation. One of his greatest triumphs was work on gravity that he developed using the data and ideas of the German astronomer Johannes Kepler and others about the motion of the planets.

Newton realized that the gravitational force F between two objects with masses M and m whose centres are separated by distance r is:

- always attractive
- proportional to $\frac{1}{r^2}$
- proportional to M and m.

This can be summed up in the equation

$$F \propto \frac{Mm}{r^2}$$

Laws that depend on $\frac{1}{r^2}$ are known as inverse-square. If the distance between the two masses is **doubled** without changing mass, then the force between the masses goes down to **one-quarter** of its original value.

To use this equation numerically a constant of proportionality is needed and is given the symbol G,

$$F = \frac{GMm}{r^2}$$

G is known as the **universal gravitational constant** and it has an accepted value of 6.67×10^{-11} N m^2 kg^{-2}. Gravity is always attractive so if the distance is measured *from* the centre of mass M *to* mass m then the force on m due to M is *towards* M. In other words, the force is in the opposite direction to the direction in which the distance is measured. You may see some books where a negative sign is introduced to predict this direction. The IB Diploma Programme physics course does not attribute negative signs to attractive forces, the responsibility of keeping track of the force direction is with you.

Nature of science

A universal constant?

What does it mean to say that G is a universal gravitational constant? Newton did not know what the size of the constant was (as the value was determined over a hundred years later by Cavendish). In a sense, it did not matter. Newton realized that all objects are attracted to the Earth – the apocryphal story of him seeing a falling apple reminds us that he knew this. The insight that Newton had was to realize that the force of gravity went on beyond the apple tree and stretched up into the sky, to the Moon, and beyond. He realized that the Moon was falling continuously towards the Earth under the influence of gravity and because it was also moving "horizontally" it was in continuous orbit.

Worked examples

1 Calculate the force of attraction between an apple of mass 100 g and the Earth.
Mass of Earth = 6.0×10^{24} kg.
Radius of Earth = 6.4 Mm

Solution

$$F = \frac{GMm}{r^2} = \frac{6.7 \times 10^{-11} \times 0.1 \times 6.0 \times 10^{24}}{(6.4 \times 10^6)^2}$$

$$= 1.1\,\text{N}$$

2 Calculate the force of attraction between a proton of mass 1.7×10^{-27} kg and an electron of mass 9.1×10^{-31} kg a when they are at a distance of 1.5×10^{-10} m apart.

Solution

$$F = \frac{GMm}{r^2}$$

$$= \frac{6.7 \times 10^{-11} \times 1.7 \times 10^{-27} \times 9.1 \times 10^{-31}}{(1.5 \times 10^{-10})^2}$$

$$= 4.6 \times 10^{-48}\,\text{N}$$

(Compare this with the electrostatic attraction of the electron and proton at this separation of about 10^{-8} N.)

Gravitational field strengths re-visited

Knowledge of Newton's law of gravitation means that the definition of gravitational field strength can be taken further to help our study of real situations.

The field strength at a distance r from a point mass, M

The simplest case is that of a single point mass M placed, as usual, a long way from any other mass. In practice we say that the point mass is at an infinite distance from any other mass because if r is very large then $\frac{1}{r^2}$ is extremely small (mathematicians say that as r tends to infinity, $\frac{1}{r^2}$ tends to zero).

As usual, the mass of our small test object is m. This means that the *magnitude* of the force F between the two masses M and m is

$$F = \frac{GMm}{r^2}$$

so that the gravitational field strength g is

$$g = \frac{F}{m} = \frac{GM}{r^2}$$

As before, the direction is measured outwards *from M* but the force on the point mass is in the opposite direction *towards M*.

The field strength at a distance r from the centre outside a sphere of mass M

It turns out that the answer for g outside a spherical planet is exactly the same as g for the point mass just quoted:

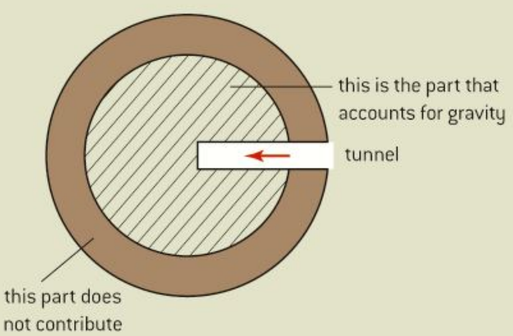
$$g = \frac{GM}{r^2}$$

If we are outside the sphere, all the mass acts as though it is a point mass of size M positioned at the centre of the sphere (the centre of mass). We only need one equation for both point masses and spheres.

Nature of science

And inside the Earth...?

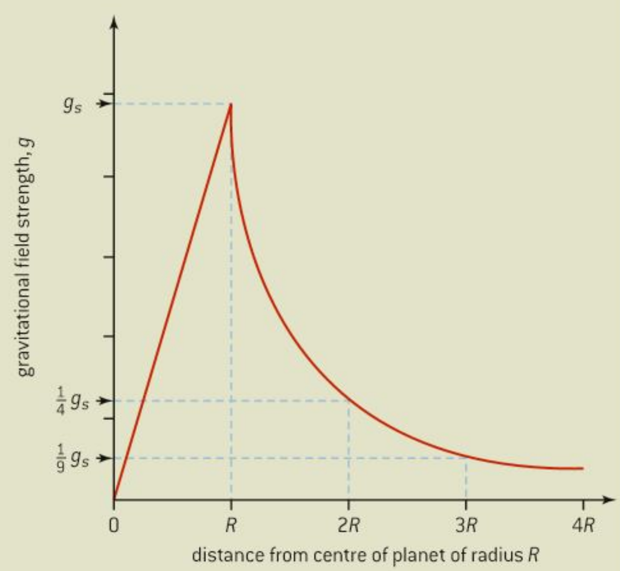
Science is all about asking questions. Outside the Earth, the gravitational field strength varies as $\frac{1}{r^2}$. But what happens to gravity inside the Earth? Is it zero? Is it a constant? Does it become larger and larger, reaching infinity, as we get closer to the centre? You might, at first sight, expect this from Newton's law.



▲ Figure 5 Journey to the centre of the Earth.

In *Journey to the centre of the Earth* the novelist Jules Verne imagined going through a volcanic tunnel to the centre of the planet. Visualize his travellers halfway down the tunnel as they stand on the surface of a "smaller" Earth defined by their present distance from the centre.

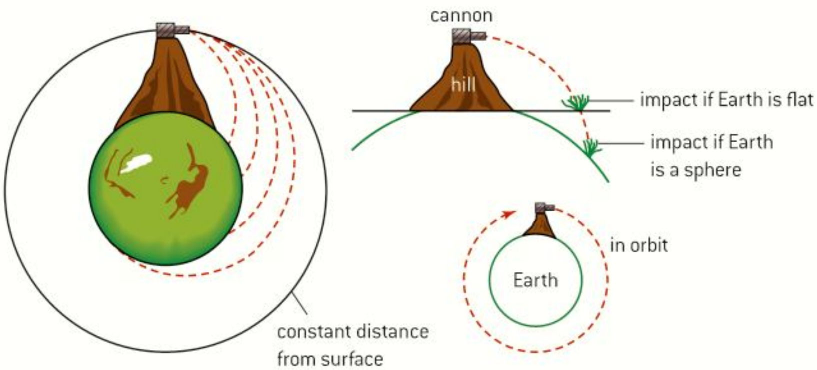
Two remarkable things happen: The first is that the contribution from the shell of Earth "above" the travellers makes no contribution to the gravitational field; all the different parts of the outer shell cancel out. The second thing that happens is that only the mass inside the "smaller" Earth contributes to the gravitational pull. The mass of this "smaller" Earth varies with r^3 assuming a constant density for the Earth. Because the gravitational force varies with $\frac{1}{r^2}$, together these give an overall dependence of r. The whole graph for the variation of g for a planet (inside and out) is given in figure 6, g_s is the gravitational field strength at the surface



▲ Figure 6 Gravitational field strength inside and outside the Earth.

Linking orbits and gravity

The gravitational force of a planet provides the centripetal force to keep a satellite in orbit.



▲ Figure 7 Newton's cannon – how he thought about orbits.

Newton had an insight into this too. He used the example of a cannon on a high mountain (figure 7). The cannon fires its cannonball horizontally and it accelerates vertically downwards. On a flat Earth, it will eventually hit the ground. Newton knew that the Earth was a sphere and that the curvature of the earth allowed the ball to travel further before hitting the ground.

He then imagined the ball being fired at larger and larger initial speeds. Eventually the shell will travel "horizontally" at such a high speed that the curvature of the Earth and the curve of the trajectory will be exactly the same. When this happens the distance between shell and surface is constant and the shell is in orbit around the Earth.

What do you expect to happen to the trajectory of the cannon ball if it is fired at even greater speeds? To check your conclusions, find an applet on the Internet that will allow you to vary the firing speed of Newton's cannon. A good starting point for the search is "applet Newton gun".

This motion of a satellite around the Earth can be analysed by combining the ideas of centripetal force and gravitational attraction. The gravitational attraction F_G provides the centripetal force F_c, so (ignoring signs)

$$F_c = F_G = \frac{mv^2}{r} = \frac{GM_E m}{r^2}$$

where M_E is the mass of the Earth, r is the distance from the satellite to the centre of the Earth, v is the linear speed of the satellite, and m is its mass.

These equations can be simplified to

$$v = \sqrt{\frac{GM_E}{r}}$$

This equation predicts the speed of the satellite at a particular radius. Notice that the speed does not depend on the mass of the satellite. All satellites travelling round the Earth at the same distance above the surface have the same speed.

As an exercise, use the equations on page 250 to show that

$$\omega^2 = \frac{GM_E}{r^3}$$

Worked examples

1 Calculate the gravitational field strength at the surface of the Moon.

(Mass of Moon $= 7.3 \times 10^{22}$ kg; radius of the Moon $= 1.7 \times 10^6$ m)

Solution

$$g = \frac{GM}{r^2} = \frac{6.7 \times 10^{-11} \times 7.3 \times 10^{22}}{(1.7 \times 10^6)^2} = 1.7 \text{ N kg}^{-1}$$

2 Calculate the gravitational field strength of the Sun at the position of the Earth.

(Mass of Sun $= 2.0 \times 10^{30}$ kg; Earth–Sun distance $= 1.5 \times 10^{11}$ m.)

Solution

$$g = \frac{GM}{r^2} = \frac{6.7 \times 10^{-11} \times 2.0 \times 10^{30}}{(1.5 \times 10^{11})^2} = 6.0 \text{ mN kg}^{-1}$$

where ω is the angular speed, and that

$$T^2 = \frac{4\pi^2 r^3}{GM_E}$$

where T is the orbital period (time for one orbit) of the satellite. This result:

(orbital period of a satellite)$^2 \propto$ *(orbital radius)*^{3}

is known as **Kepler's third law**. It is one of the three laws that he discovered when he analysed Tycho Brahe's data.

Worked examples

1 Calculate the orbital period of Jupiter about the Sun. Mass of Sun $= 2.0 \times 10^{30}$ kg; radius of Jupiter's orbit $= 7.8 \times 10^{11}$ m)

Solution

$$T^2 = \frac{4\pi^2 r^3}{GM_S}$$

$T = 3.7 \times 10^8$ s (about 12 years)

2 The orbital time period of the Earth about the Sun is 3.2×10^7 s. Calculate the orbital period of Mars.

(radius of Earth orbit $= 1.5 \times 10^{11}$ m; radius of Mars orbit $= 2.3 \times 10^{11}$ m)

Solution

$$\frac{T_M^2}{T_E^2} = \frac{r_M^3}{r_E^3}$$

$$T_M = T_E\sqrt{\frac{r_M^3}{r_E^3}} = 6.1 \times 10^7 \text{ s (about 1.9 years)}$$

Nature of science

What Newton knew ...

Several decades before Newton began working on his ideas of gravitation, Kepler had used his own and others' astronomical data to show that, for the planets,

(radius of the orbit)$^3 \propto$ *(time period of orbit)*^{2}

This was an *empirical* result (meaning that it came from experimental data).

Newton was able to show that this proportionality arose from his own theory; this was a *theoretical* result (meaning that it came from a theory, not data).

Here are some data for the satellites of Jupiter.

Moon	Distance from centre of Jupiter/10^3 km	Orbital period/ days
Io	420	1.8
Europa	670	3.6
Ganymede	1070	7.2
Callisto	1890	16.7

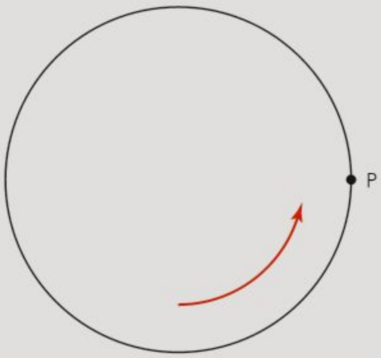
Use these data to show that

(radius of the orbit)$^3 \propto$ (time period of orbit)^2

Or, put another way, that $\frac{\text{(radius of the orbit)}^3}{\text{(time period of orbit)}^2}$ is a constant.

Questions

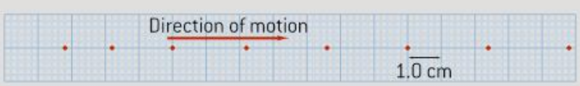
1 A particle P is moving in a circle with uniform speed. Draw a diagram to show the direction of the acceleration a and velocity v of the particle at one instant of time.



2 State what provides the centripetal force that causes a car to go round a bend.

3 State the centripetal force that acts on a particle of mass m when it is travelling with linear speed v along the arc of a circle of radius r.

4 (*IB*) At time $t = 0$ a car moves off from rest in a straight line. Oil drips from the engine of the car with one drop every 0.80 s. The position of the oil drops on the road are drawn to scale on the grid below such that 1.0 cm represents 4.0 m. The grid starts at time $t = 0$.



a) (i) State the feature of the diagram that indicates that the car accelerates at the start of the motion.

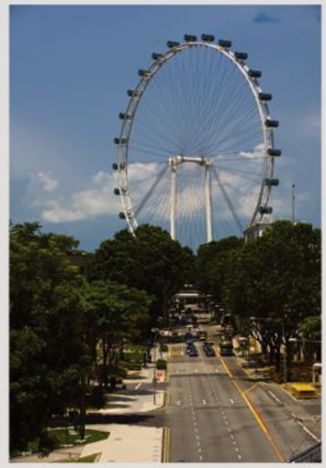
(ii) Determine the distance moved by the car during the first 5.6 s of its motion.

b) The car then turns a corner at constant speed. Passengers in the car who were sitting upright feel as if their upper bodies are being "thrown outwards".

(i) Identify the force acting on the car, and its line of action, that enables the car to turn the corner.

(ii) Explain why the passengers feel as if they are being thrown outwards.

5 The Singapore Flyer is a large Ferris wheel of radius 85 m that rotates once every 30 minutes.



a) Calculate the linear speed of a point on the rim of the wheel of the Flyer.

b) (i) Determine the fractional change in the weight of a passenger on the Flyer at the top of the ride.

(ii) Explain whether the passenger has a larger or smaller apparent weight at the top of the ride.

c) The capsules need to rotate to keep the floor of the cabin in the correct place.

Calculate the angular speed of the capsule about its central axis.

6 The radius of the Earth is 6400 km. Determine the linear speed of a point on the ground at the following places on Earth:

a) Quito in Ecuador (14 minutes of arc south of the Equator)

b) Geneva in Switzerland (46° north of the Equator)

c) the South Pole.

7 A school bus of total mass 6500 kg is carrying some children to school.

a) During the journey the bus needs to travel round in a horizontal curve of radius 150 m. The dynamic coefficient of friction between the tyres and the road surface is 0.7. Estimate the maximum speed at which the driver should attempt the turn.

b) Later in the journey the driver needs to drive across a curved bridge with a radius of curvature of 75 m. Estimate the maximum speed if the bus is to remain in contact with the road.

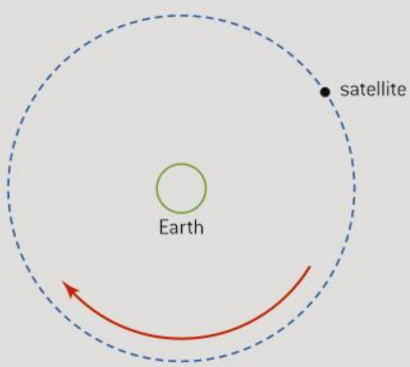
8 A velodrome used for bicycle races has a banking angle that varies continuously from 0° to 60°. Explain how the racing cyclists use this variation in angle to their advantage in a race.

Data needed for these questions:

Radius of Earth = 6.4 Mm;
Mass of Earth = 6.0×10^{24} kg;
Mass of Moon = 7.3×10^{22} kg;
Mass of Sun = 2.0×10^{30} kg;
Earth–Moon distance = 3.8×10^{8} m;
Sun–Earth distance = 1.5×10^{11} m;
$G = 6.67 \times 10^{-11}$ N m^2 kg^{-2}

9 Deduce how the radius R of the circular orbit of a planet around a star of mass m_s relates to the period T of the orbit.

10 A satellite orbits the Earth at constant speed as shown below.



a) Explain why, although the speed of the satellite is constant, the satellite is accelerating.

b) Discuss whether or not the gravitational force does work on the satellite.

11 Determine the distance from the centre of the Earth to the point at which the gravitational field strength of the Earth equals that of the Moon.

12 The ocean tides on the Earth are caused by the tidal attraction of the Moon and the Sun on the water in the oceans.

a) Calculate the force that acts on 1 kg of water at the surface of the sea due its attraction by the

(i) Moon

(ii) Sun.

b) *Optional – difficult.* Explain why there are two tides every day at many coastal points on the Earth.

[Hint: there are two parts to the answer, why a tide at all, and why two every day.]

13 There are two types of communication satellite. One type of communication satellite orbits over the poles at a distance from the centre of the Earth of 7400 km; the other type is geostationary with an orbital radius of 36 000 km. Geostationary satellites stay above one point on the equator whereas polar-orbit satellites have an orbital time of 100 minutes.

Calculate:

a) the gravitational field strength at the position of the polar-orbit satellite

b) the angular speed of a satellite in geostationary orbit

c) the centripetal force acting on a geostationary satellite of mass 1.8×10^3 kg.