B ENGINEERING PHYSICS

Introduction

Engineering physics covers some of the key topics that you would expect to meet on an undergraduate engineering course. These are quite diverse topics covering rotational motion, thermodynamics, fluid mechanics, and forced vibrations and resonance. The mathematical content of these topics is naturally restricted to be in line with a pre-university course; however, those students with a strong mathematical inclination will find much in these sub-topics to stimulate further research.

B.1 Rigid bodies and rotational dynamics

Understanding

- Torque
- Moment of inertia
- Rotational and translational equilibrium
- Angular acceleration
- Equations of rotational motion for uniform angular acceleration
- Newton's second law applied to angular motion
- Conservation of angular momentum

Nature of science

Extended objects

The work covered until now has almost invariably used the model of an object being represented by a single particle of zero dimensions. This model works well for many situations but falls short when forces are not applied along a line passing through the centre of mass of the object. In such cases an extended object can still be modelled; however, objects of different shape will not always move in the same way when forces are applied to them. Using simplifying models, engineers were able to design and manufacture sophisticated machines that had significant impact during the industrial revolution of the late eighteenth to early nineteenth century.

Applications and skills

- Calculating torque for single forces and couples
- Solving problems involving moment of inertia, torque, and angular acceleration
- Solving problems in which objects are in both rotational and translational equilibrium
- Solving problems using rotational quantities analogous to linear quantities
- Sketching and interpreting graphs of rotational motion
- Solving problems involving rolling without slipping

Equations

- Torque equation: $\Gamma = Fr\sin\theta$
- Moment of inertia: $I = \sum mr^2$
- *Newton's second law for rotational motion:* $\Gamma = I\alpha$
- Relationship between angular frequency and frequency: $\omega = 2\pi f$
- Equations of motion for constant angular acceleration:
- $\omega_f = \omega_i + \alpha t$
- $\omega_f^2 = \omega_i^2 + 2\alpha\theta$
- $\theta = \omega_i t + \frac{1}{2}\alpha t^2$
- Angular momentum: $L = I\omega$
- Rotational kinetic energy: $E_{K_{rot}} = \frac{1}{2} I\omega^2$

> **Note**
> - The angular acceleration is different from both centripetal acceleration (a_c) and tangential acceleration (a_t) – this is the rate of change of the linear tangential velocity with time.
> - As an object moving in a circle undergoes angular acceleration, its tangential acceleration increases as does its centripetal acceleration.
> - The tangential acceleration of the body is related to the angular acceleration and the radius of the circle (r) by the relationship:
> $a_t = r\alpha$
>
> This is because $v_t = \omega r$
>
> This means that $\alpha = \frac{\Delta v}{r\Delta t} = \frac{a_t}{r}$ where v_t is the tangential velocity of the object. Thus the linear quantity (velocity or acceleration) is the angular quantity multiplied by the radius.

Introduction

This sub-topic is very closely related to the linear mechanics and circular motion that you studied in Topics 2 and 6. There are quantities that are directly analogous to the linear quantities of displacement, velocity, acceleration, force, etc. With the knowledge that you have already absorbed and an understanding that a rigid body is an extension of a point object you will be able to solve most rotational dynamics problems.

Uniform motion in a circle

You will remember that for a body moving around a circle with a constant linear speed (or angular speed) there needs to be a means of providing a centripetal force. This can be a contact force, such as tension or friction (or a component of these forces), or a force provided by a field, such as Newton's law of gravitation or Coulomb's law.

In all cases the force (or component) providing the centripetal force will be given by:

$$F = \frac{mv^2}{r} = m\omega^2 r$$

These terms were defined in Topic 6. Remember that ω is known as the angular velocity or angular frequency and is related to frequency f by the equation:

$$\omega = 2\pi f$$

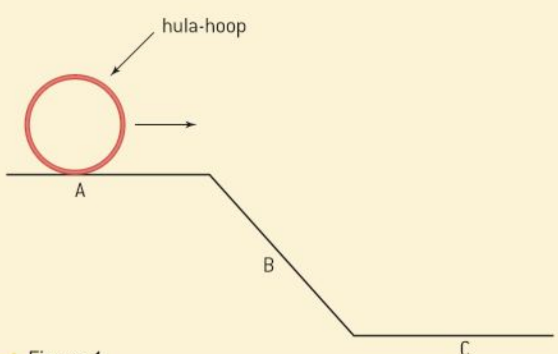
Angular acceleration

Let us consider an object moving in the circle so that it is given an angular acceleration and its angular speed increases. We define **angular acceleration (α) as the rate of change of angular speed with time.**

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

α is measured in radian per second squared (rad s^{-2})

Worked example



▲ Figure 1

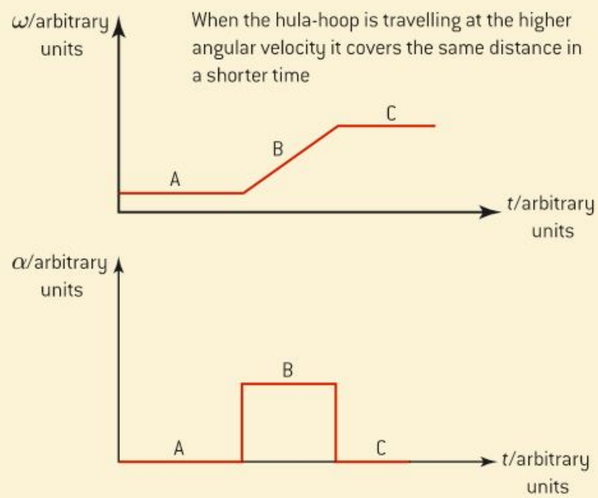
The diagram shows a hula-hoop (a large plastic ring) rolling with constant angular speed along a horizontal surface prior to rolling down a uniform inclined plane. When it reaches a second horizontal surface it, again, moves with a constant angular speed.

Sketch graphs to show the variation with time of a) the angular velocity and b) the angular acceleration of the hula-hoop.

Solution

a) The graph shows the hula-hoop travelling with constant angular velocity along A and C. It has a greater value along C since it has now undergone angular acceleration. As B is of constant gradient, the angular acceleration is constant here.

b) The second graph shows zero angular acceleration throughout A and C and a constant angular acceleration along B. *You should compare these graphs with those for a point object moving along a frictionless surface.*



▲ Figure 2

Equations of motion

In Topic 2 we met the four equations of motion under constant linear acceleration:

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{(v + u)t}{2}$$

In rotational dynamics there are four analogous equations that apply to a body moving with constant angular acceleration:

$$\omega_f = \omega_i + \alpha t$$

$$\theta = \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha\theta$$

$$\theta = \frac{(\omega_i + \omega_f)t}{2}$$

Here ω_i = initial velocity (in rad s^{-1}), ω_f = final velocity (in rad s^{-1}), θ = angular displacement (in rad) or sometimes just "angle", α = angular acceleration (in rad s^{-2}), t = time taken for the change of angular speed (in s).

Note

The IB Physics Data Booklet does not include the last of these rotational equations – it is probably the most straightforward of the equations because it simply equates two ways of expressing the average angular speed.

Worked example

A wheel is rotated from rest with an angular acceleration of 8.0 rad s^{-2}. It accelerates for 5.0 s.

Determine:

a) the angular speed

b) the number of revolutions that the wheel has rotated through.

Solution

a) $\omega_f = \omega_i + \alpha t \Rightarrow \omega_f = 0 + 8.0 \times 5.0$
$= 40.0$ rad s^{-1}

b) $\theta = \frac{(\omega_i + \omega_f)t}{2} = \frac{(0 + 40.0)5.0}{2} = 100$ rad

Each revolution makes an angle of 2π radian so the number of revolutions $= \frac{100}{2\pi}$

$=15.9$ revolutions.

▲ Figure 3 A pair of flywheels with much of the mass distributed to be around the perimeter.

Moment of inertia

The moment of inertia ,I, of a body is the rotational equivalent of the role played by mass in linear dynamics. In a similar way to the inertial mass of an object being a measure of its opposition to a change in its linear motion, the moment of inertia of an object is its resistance to a change in its rotational motion. Objects such as flywheels that need to retain their rotational kinetic energy are designed to have large moments of inertia as shown in figure 3.

The moment of inertia of an object depends on the axis about which it is rotated.

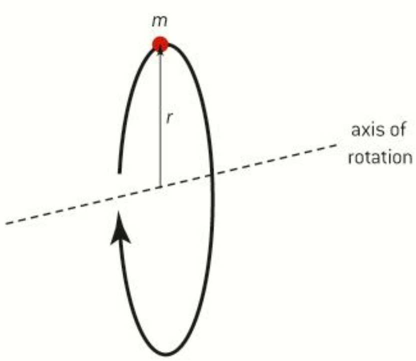
For a particle (a single point) of mass m rotating at a distance r about an axis, the moment of inertia is given by:

$$I = mr^2$$

Moments of inertia are scalar quantities and are measured in units of kilogram metres squared (kg m^2).

For an object consisting of more than one point mass, the moment of inertia about a given axis can be calculated by adding the moments of inertia for each point mass.

$$I = \sum mr^2$$



▲ Figure 4 Moment of inertia of a point mass.

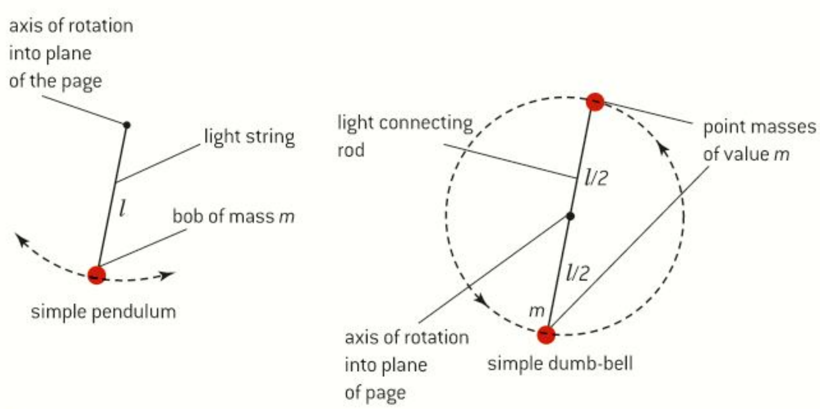
This summation is a mathematical abbreviation for $m_1r_1^{\,2} + m_2r_2^{\,2} + m_3r_1^{\,3} + \ldots + m_nr_n^{\,2}$ when the object is made up of n point masses.

The moment of inertia of a simple pendulum of length l and mass m is given by

$$I_{\text{pendulum}} = ml^2$$

while that of a simple dumb-bell consisting of two masses m connected by a light rod of length l, when rotating about the centre of the rod, will be

$$I_{\text{dumb-bell}} = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = 2m\left(\frac{l}{2}\right)^2 = \frac{1}{2}ml^2$$



▲ Figure 5 Moment of inertias of a simple pendulum and dumb-bell.

In each of these cases we assume that the mass is a point mass and that the string and the connecting rod are so light that their masses can be ignored.

For a more complicated structure we need to use integral calculus in order to derive formulae for moments of inertia. In IB Diploma Programme physics questions, any equations that need to be used for moments of inertia will be provided in the question.

Note

With linear motion there is just one "single" mass value for an object. In rotational motion, the moment of inertia is also a function of the position of the axis of rotation. This means that there are an infinite range of possible moments of inertia for any one object.

Torque

Torque ,Γ, can be defined as shown in figure 6. Consider force F acting at point P on an object. The direction of F is such that it makes an angle θ to the radius of the circle in which the object rotates. The torque will be given by:

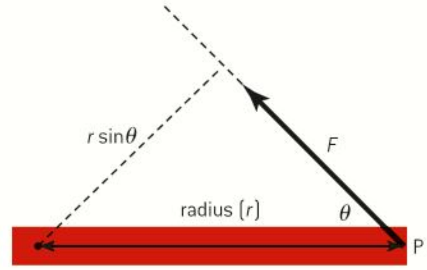
$$\Gamma = Fr\sin\theta$$

This is sometime called the force multiplied by the "arm of the lever".

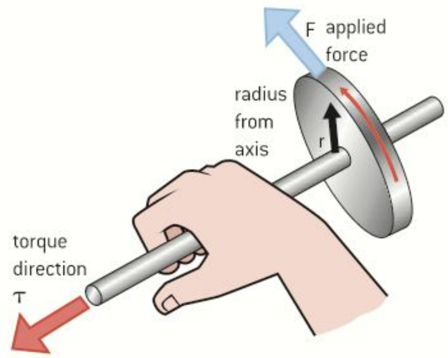
Torques are also called "moments", but to avoid confusion between "moment" and "momentum" we advise sticking to the term torque. They are (pseudo) vector quantities with the direction perpendicular to the plane of the circle in which the object rotates. Imagine your right hand gripping the axis about which the object rotates so that the fingers curl round the axis, when your thumb is "up" it will be pointing in the direction of the torque (as shown in figure 7). This rule also applies to the directions of angular displacement ,θ, angular velocity ,ω, and angular acceleration ,α, which, like torque, are all vector quantities.

The maximum torque that a force can apply to a body is when the force is perpendicular to the arm of the lever. In this case $\theta = 90°$ and so $\sin\theta = 1$. This means that

$$\Gamma = Fr$$



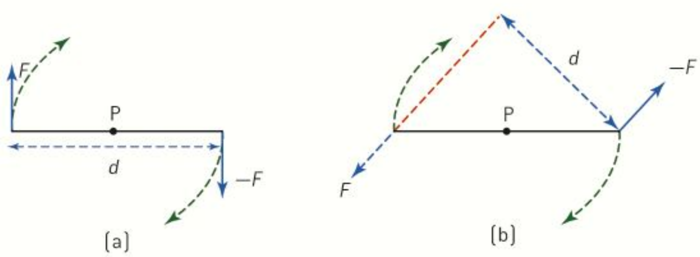
▲ Figure 6 Definition of torque.



▲ Figure 7 Direction of torque.

Couples

A **couple consists of a pair of equal and opposite forces that do not act in the same straight line** (see figure 7). This combination of forces produces a torque that causes an object to undergo angular acceleration without having any translational acceleration.



▲ Figure 8 Two examples of couples.

The torque of the couple about the pivot P in both figure 8(a) and (b) is equal to Fd (that is the product of one of the forces and the perpendicular distance between them).

- A single resultant force acting through the centre of mass of an object will produce translational acceleration. This means that the object will move in a straight line. An object in free-fall is an example of this as (ignoring air resistance) the pull of gravity is the only force acting on it.
- The same force, but acting through a point displaced from the centre of mass of the object, will produce a combination of both linear and angular acceleration. This means that the object will move in a helical path.

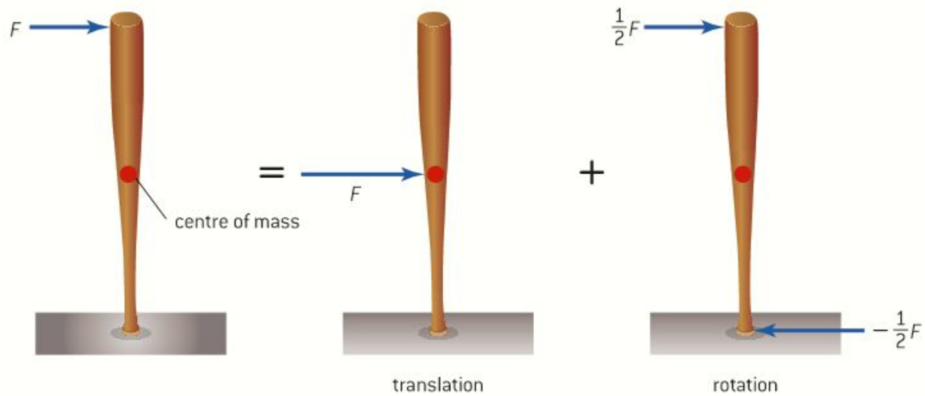
Note

Torque is measured in units of newton metre (N m) but don't confuse this with the unit of work or energy (the joule). In the case of torque the force is perpendicular to the direction in which the object moves but in the case of work the force and direction moved are the same. Torque is a vector quantity while work is a scalar.

Figure 9 shows a force acting at the top of an object – the effect of this force can be split into the (i) same force acting at the centre of mass of the object (producing translation) and (ii) a couple (producing rotation). The couple consists of $\frac{1}{2}F$ at the top and $-\frac{1}{2}F$ at the bottom of the object.

This is a general principle that can be applied to any force not acting through the centre of mass of an object.

A couple acting on an object will produce angular acceleration with no linear motion at all. This means that the object will move in a circular path (as shown in figure 8).



▲ Figure 9 Force acting through point displaced from centre of mass.

Newton's first law for angular motion – rotational equilibrium

We saw in Topic 2 that a body in (translational) equilibrium does not accelerate but remains in a state of rest or travels with uniform velocity. This means that there can be no resultant force acting on the body in line with Newton's first law of motion. For an object to be in rotational equilibrium there can be no external resultant torque acting on it – it will then remain in a state of rest or continue to rotate with constant angular velocity.

For rotational motion Newton's first law may be stated as:

An object continues to remain stationary or to move at a constant angular velocity unless an external torque acts on it.

When the object is in rotational equilibrium:

total clockwise torque = total anticlockwise torque.

This statement is often called the "principle of moments".

Newton's second law for angular motion – angular acceleration

Again, in Topic 2 we considered the simpler statement of Newton's second law of (translational) motion equating force to the product of mass and acceleration. The rotational equivalent of Newton's second law relates the angular acceleration and torque on a body of moment of inertia.

$$\Gamma = I\alpha$$

In IB Diploma Programme Physics questions, the axis to which the torque is applied and the axis about which the moment of inertia is taken will always be the same (although there are quite simple ways of dealing with situations when this is not the case).

Newton's third law for rotational motion

You can probably guess that this version of the law says that action torque and reaction torque are equal and opposite – this pair of torques, like action and reaction forces for linear motion, act on different bodies. If body A applies a torque to body B, then body B applies an equal and opposite torque to body A.

Angular momentum

Linear momentum (p) is a vector quantity defined as being the product of mass and velocity. Angular momentum (L) is the rotational equivalent to this and is defined as being the product of a body's moment of inertia and its angular velocity; this, too, is a vector.

$$L = I\omega$$

Angular momentum is measured in units of kg m^2 rad s^{-1}.

Worked example

A couple, consisting of two 4.0 N forces, acts tangentially on a wheel of diameter 0.60 m. The wheel starts from rest and makes one complete rotation in 2.0 s. Calculate:

a) the angular acceleration

b) the moment of inertia of the wheel.

Solution

a) $\theta = \omega_i t + \frac{1}{2}\alpha t^2 \Rightarrow 2\pi = 0 + \frac{1}{2}\alpha \times 2.0^2$

$\alpha = \frac{4\pi}{4.0} = \pi = 3.14 \text{ rad s}^{-2}$

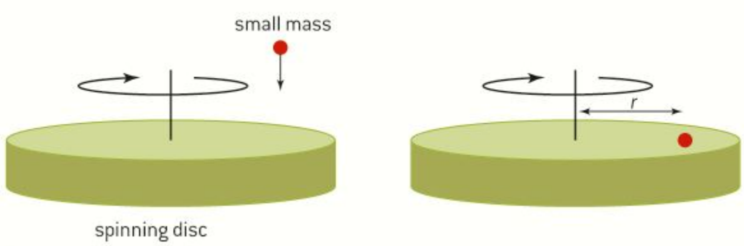
b) torque of couple $\Gamma = Fd = 4.0 \times 0.60 = 2.4$ N m

as $\Gamma = I\alpha \Rightarrow I = \frac{\Gamma}{\alpha} = \frac{2.4}{3.14} = 0.76$ kg m^2

The conservation of angular momentum

In linear dynamics we find the total (linear) momentum of a system remains constant providing no external forces act on it. In rotational dynamics, the total angular momentum of a system remains constant providing no external torque acts on it.

Figure 10 shows a small mass being gently dropped onto a freely spinning disc. The addition of the mass increases the combined moment of inertia of the disc and the mass and so the angular velocity of the system now falls in order to conserve the angular momentum.



▲ Figure 10 An example of the conservation of angular momentum.

The conservation of angular momentum has many applications in physics, such as the speeding up of an ice skater when pirouetting with decreasing angular momentum, or a gymnast putting in turns or twists by changing the distribution of mass.

Rotational kinetic energy

You probably feel confident about the analogies between linear and rotational dynamics, so it should not surprise you that the relationship for angular kinetic energy is

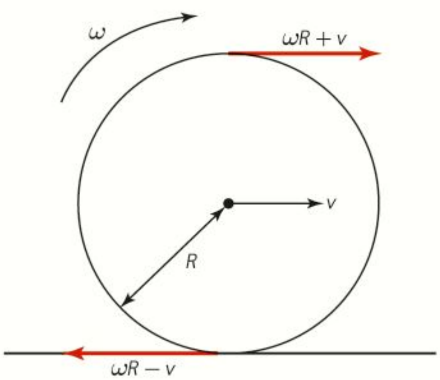
$$E_{K_{rot}} = \frac{1}{2} I\omega^2$$

This, like all energies, is a scalar quantity and could, in principle, be found by adding the (translational) kinetic energies of all the particles making up a rotating object.

Two more useful rotational analogies are given in the table below.

Quantity	Linear equation	Angular equation
work	$W = Fs$	$W = \Gamma\theta$
power	$P = Fv$	$P = \Gamma\omega$

Rolling and sliding



▲ Figure 11 Disc rolling on a flat surface.

If an object makes a perfectly frictionless contact with a surface it is impossible for the object to roll – it simply slides. When there is friction the object can roll; as the point of contact between the rolling body and the surface along which it rolls is instanteously stationary, the coefficient of static friction should be used in calculations involving rolling. The point of contact must be stationary because it does not slide. Figure 11 shows a disc of radius R rolling along a flat surface such that its centre of mass has a velocity v. Each point on the perimeter of the disc will have a tangential velocity $= \omega R$.

This means that the top of the disc will have total velocity of $\omega R + v$ and the point of contact with the surface will have a velocity of $\omega R - v$. As the disc is not slipping, the bottom of the disc has zero instantaneous velocity and so $\omega R = v$. This means that the top of the disc will have an instantaneous velocity $= 2v$.

The total kinetic energy of a body that is rolling without slipping will be $= \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$. When an object rolls down a slope so that it loses a vertical height h, the loss of gravitational potential energy will become the total kinetic energy. This gives us

$$mgh = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

Worked example

A solid ball, of radius of 45 mm, rolls down an inclined plane of length 2.5 m. The sphere takes a time of 6.0 s to roll down the plane. Assume that the ball does not slip.

The moment of inertia, I, of a solid ball of mass M and radius R is given by $I = \frac{2}{5} MR^2$

a) Calculate the velocity of the sphere as it reaches the end of the plane.

b) Calculate the angular velocity of the sphere as it reaches the end of the plane.

c) Determine the angle of inclination of the plane.

d) Comment on whether the assumption that the ball does not slip was appropriate in this instance.

Solution

a) For the ball starting at rest rolling along the slope for time t the (translational) equations of motion give:

$$s = \frac{1}{2}at^2 \text{ and } v = at$$

Substituting the values for s and t

$$a = \frac{2s}{t^2} = \frac{2 \times 2.5}{6.0^2} = 0.14 \text{ m s}^{-2}$$
$$v = at = 0.14 \times 6.0 = 0.84 \text{ m s}^{-1}$$

b) The angular speed $\omega = \frac{v}{r} = \frac{0.84}{45 \times 10^{-3}} = 18.7 \approx 19 \text{ rad s}^{-1}$

c) from the equations $mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2$ and $v = \omega R$

$$mgh = \frac{1}{2}v^2\left(\frac{I}{R^2} + m\right)$$

substituting for the moment of inertia equation we have:

$$Mgh = \frac{1}{2}v^2\left(\frac{\frac{2}{5}MR^2}{R^2} + M\right)$$

Cancelling M and R^2 gives

$$gh = \frac{1}{2}v^2\left(\frac{2}{5} + 1\right) = \frac{7}{10}v^2$$

So $h = \frac{7}{10} \times \frac{v^2}{g}$ where h is the height of the end of the inclined plane.

$$h = \frac{7}{10} \times \frac{0.84^2}{9.81} = 0.05 \text{ m}$$

$$\sin\theta = \frac{0.05}{2.5} \quad \therefore \quad \theta = \sin^{-1}\left(\frac{0.05}{2.5}\right) = 1.1^\circ$$

d) This angle is very small and so there is no problem with assuming that the ball rolls without slipping.

Investigate!

Measuring the moment of inertia of a flywheel

A flywheel is mounted so that it can rotate on a horizontal axle (the axle forming part of the flywheel). A mass, suspended by a string, is wound round the axle of a flywheel ensuring that it does not overlap. When the mass is released, the gravitational potential energy of mass is converted into the linear kinetic energy of the mass + the rotational kinetic energy of the flywheel + work done against the frictional forces (all these values are taken when the string loses contact with the axle).

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + n_1W$$

The symbols here have their usual meaning with n_1 being the number of turns of string and W the work done against frictional forces during each revolution. When the string disengages, the rotational kinetic energy makes a further n_2 rotations before it comes to rest – so the rotational kinetic energy of the flywheel must be equal to n_2W or

$$\frac{1}{2}I\omega^2 = n_2W \text{ making } W = \frac{I\omega^2}{2n_2}$$

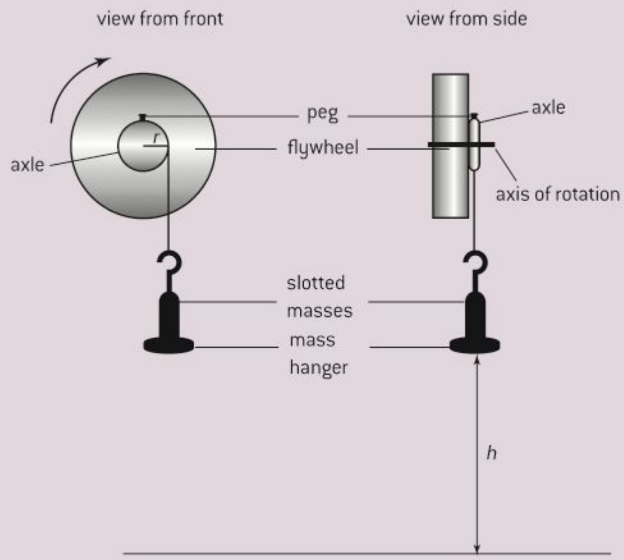
This means $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{n_1 I\omega^2}{2n_2}$

or $\quad mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2\left(1 + \frac{n_1}{n_2}\right)$

In addition to this $\omega = vr$ and $\frac{v}{2} = \frac{h}{t}$

- Measure the radius r of the axle of the system using vernier or digital callipers.

- Attach a mass hanger to one end of the piece of string and make a loop at the other end so that, as the mass hanger just touches the floor, the loop slips off the peg in the axle.
- Loop the string over the peg, wind up the string for a whole number of revolutions n_1 and measure the height h of hanger above the floor. Make sure the windings do not overlap.



▲ Figure 12 Measuring the moment of inertia of a flywheel.

- Adjust the number of slotted masses on the hanger.
- Measure the time t from the moment of release until the hanger reaches the floor.
- Count the number of revolutions n_2 the flywheel makes after the string comes off the peg until it comes to rest.
- Repeat several times and calculate mean values from which a value for the moment of inertia of the flywheel and axle I can be deduced.
- Consider how the data could be used graphically to find a value for I.